Representaciones inducidas

lunes, 29 de noviembre de 2021
11:01 a. m.

Si G es un grupo finito y $H \leq G$

$\rho: H \longrightarrow Gl(V)$

$\mathrm{Ind}_H^G V = \mathbb{K}[G] \otimes_{\mathbb{K}[H]} V \curvearrowright G$

$\mathbb{K}(G)$ es el $\mathbb{K}$-esp vet con base G

Si $\rho: G \longrightarrow Gl(V)$ $\quad H \xhookrightarrow{i} G$

$\mathrm{Res}_H^G V = V \qquad \rho \circ i: H \longrightarrow Gl(V)$

$\mathrm{Hom}_G(W, \mathrm{Ind} V) = \mathrm{Hom}_H(\mathrm{Res}\, W, V)$

Sea G un grupo de Lie y $H \leq G$ cerrado

$\rho: H \longrightarrow GL(V)$

$\underline{\mathrm{Ind}_H^G V} = \mathrm{Map}_H(G, V) = \{ f: G \longrightarrow V \mid f(gh^{-1}) = hf(g) \ \forall h \in H \text{ y } g \in G$

$G \curvearrowright \mathrm{Ind}_H^G V \qquad (g \cdot f)(v) = f(\rho(g^{-1})v)$

$\varphi: G \longrightarrow GL(V) \Rightarrow \mathrm{Res}_H^G \rho: H \longrightarrow GL(V)$

Otra construcción de $\mathrm{Ind}_H^G V$

$$\begin{tikzcd} V \arrow[r, leftrightarrow] & G\times V/\sim \arrow[d, "p=pr_1"] \\ & G/H \arrow[u, bend right, "id"'] \end{tikzcd}$$

$(g,v)\sim(gh^{-1}, h\cdot v)$ $\quad h\in H$

G/H ← variedad suave

$$\underset{\underset{\mathrm{Ind}_H^G V}{\cong}}{\Gamma(G\times V/\sim)} = \{ s: G/H \to G\times V/\sim \mid p\circ s = id \}$$ secciones de $G\times H/\sim$

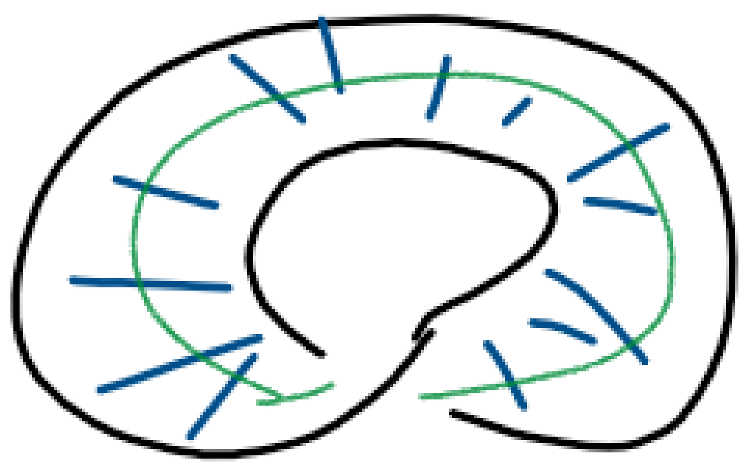
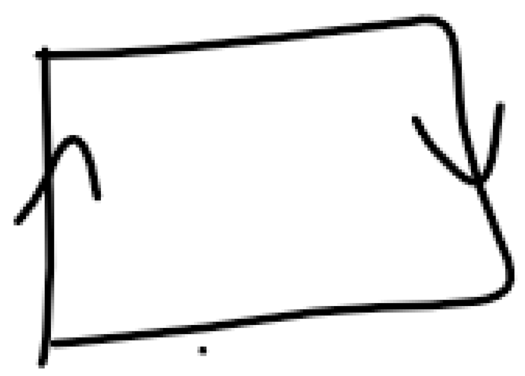
Def. Un haz vectorial es una función continua tal que satisface cierta trivialidad local y cada fibra tiene la estructura de un $\mathbb{K}$ esp. ve

$$\begin{tikzcd} \text{fibra} \quad F \arrow[r, hook] \arrow[d, phantom, "\parallel"] & E \arrow[d, "p"] & \leftarrow \text{total} \\ p^{-1}(x) & x \in B & \leftarrow \text{base} \end{tikzcd}$$

$\exists\, U \subset B \quad x \in U$ y tal que

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

Ejemplos:

•) $\mathbb{R} \longrightarrow$  $\cong$ 

$\downarrow$

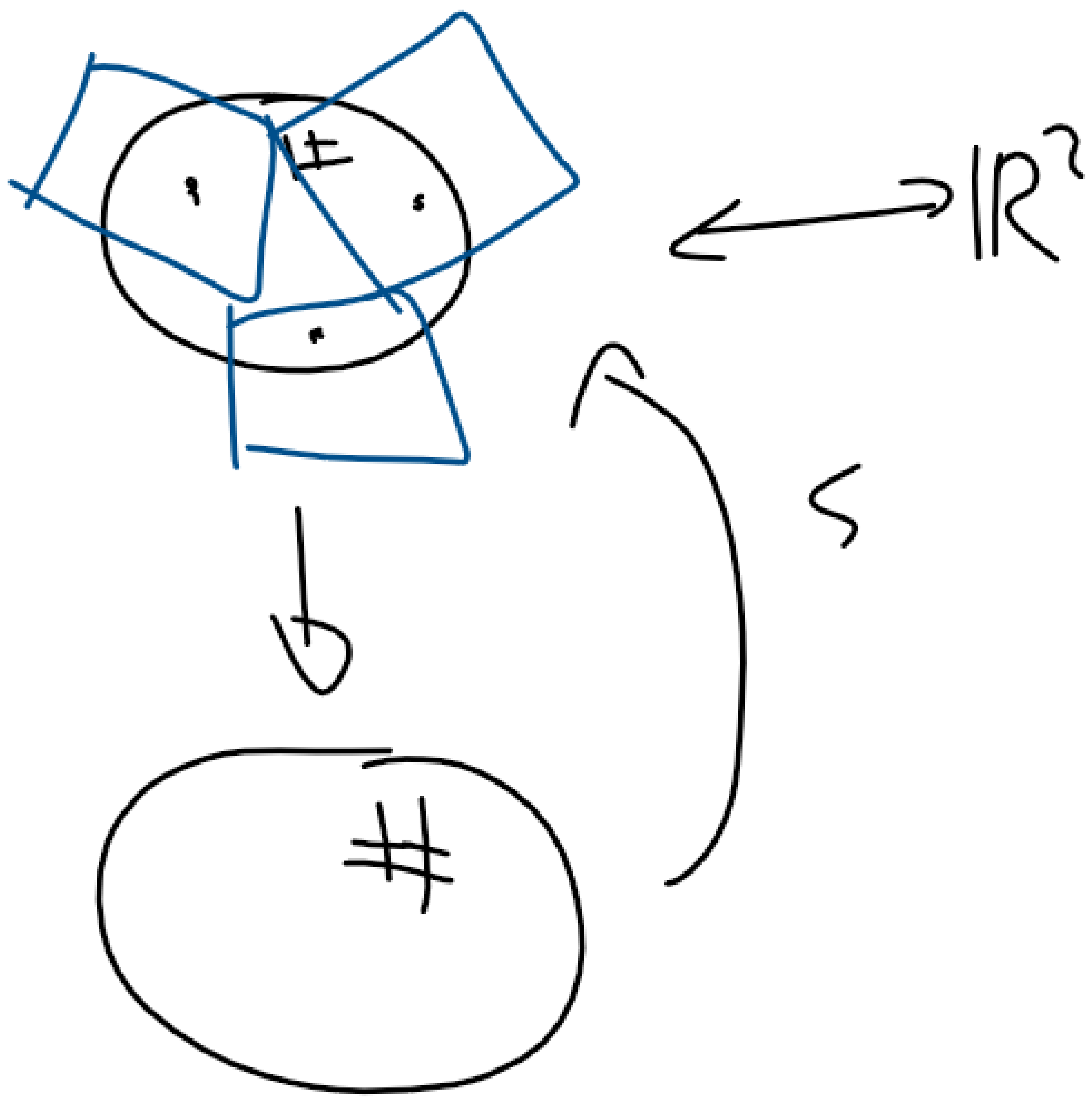
S^1

*) M^n una variedad

$$\begin{array}{c} \mathbb{R}^n \longrightarrow TM := \{(x,v) \in M, \mathbb{R}^n \mid v \in T_xM \\ \downarrow \\ M \end{array}$$

Haz tangente

$M = S^2$



$Vect(M)$

Teorema. Existe un isomorfismo de G rep.

$$\Gamma(G \times_H V) \xrightarrow{\varphi} \underline{\mathrm{Ind}_H^G V}$$

$$g \cdot s(v) = s(g^{-1}v)$$

Dem

sea $f \in \mathrm{Ind}_H^G V = \{ f : G \to V \mid \underline{f(gh^{-1}) = h f(g)} \ h \in H \ g \in G \}$

$$\Gamma \ni s_f : G/H \longrightarrow G \times_H V$$

$$gH \longmapsto (g, \underline{f(g)})$$

$$(gh, f(gh)) = (gh, h^{-1} f(g)) \sim (g, f(g))$$

Corolario. Existe un isomorfismo de G rep

$$\mathrm{Vet}(G/H) \xrightarrow{\cong} \mathrm{Ind}_H^G T_{[H]}$$

$T_{[H]}$ es el espacio tangente a G/H en $[H]$

$T_{[H]} := T_{[H]} G/H$

Dem.

$$\Gamma(TG/H) = \mathrm{Vect}(G/H)$$

$$\Gamma(\underbrace{TG/H}_{} \to G/H) \xrightarrow{\varrho} \mathrm{Ind}_H^G T_{[H]}$$

$$TG/H \longrightarrow G \times_H T_{[H]}$$

$$(x, v) \longmapsto (x, x^{-1}v)$$

$$(xH, xv) \longleftarrow (x, v)$$

$$x^{-1} : G/H \longrightarrow G/H$$

$$dx^{-1} : T_{[H]} \longrightarrow T_{[H]}$$

Teorema (Reciprocidad de Frobenius). Existe un isomorfismo

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G E) \longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G V, E)$$

Dem.

$$ev_1 : \mathrm{Ind}_H^G E \to E$$

$$f \longmapsto ev_1 \circ f$$

$$F(v)(g) = f(g^{-1}v) \qquad F \longleftarrow f$$

Proposición. Existe un isomorfismo

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$$

$$\Gamma(G \times_H V_{\mathbb{C}}) \longrightarrow \widehat{\bigoplus} \rho \otimes (\bar{\rho} \otimes V_{\mathbb{C}})^H$$

Dem

$$\Gamma(G \times_H V_{\mathbb{C}}) \cong \mathrm{Ind}_H^G V_{\mathbb{C}} = \widehat{\bigoplus_{\substack{\rho \text{ rep irr.} \\ \text{de } G}}} \rho \otimes \underbrace{\mathrm{Hom}_G(\bar{\rho}, \mathrm{Ind}_H^G V_{\mathbb{C}})}_{\text{parte isotípica}}$$

$$= \widehat{\bigoplus_{\rho}} \rho \otimes \mathrm{Hom}_H(\mathrm{Res}\,\bar{\rho}, V_{\mathbb{C}}) \qquad (\mathrm{Res}\,\bar{\rho} = \bar{\rho})$$

$$= \widehat{\bigoplus_{\rho}} \rho \otimes (\bar{\rho} \otimes V_{\mathbb{C}})^H$$

$G \curvearrowright V$

$$V = V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}$$

$$\mathrm{Hom}(V^*, W) \cong V \otimes W$$

Ejemplo:

$G = O_3$, $H = O_2$ $\qquad$ H_k son las rep. irr de G

$$\mathrm{Vect}(S^2) = \mathrm{Vect}(O_3/O_2) = \mathrm{Ind}_{O_2}^{O_3} T_{(O_2)}\mathbb{C} = \widehat{\bigoplus_{\rho}} \rho \otimes (\bar{\rho} \otimes T_{\mathbb{C}})^{O_2}$$

$$= \widehat{\bigoplus_{k \geq 0}} H_k \otimes (\bar{H}_k \otimes \mathbb{C}^2)^{O_2}$$

$$(H_0 \otimes \mathbb{C}^2)^{O_2} = 0$$

si $k > 0$ $\quad (H_0 \otimes \mathbb{C}^2)^{O_2} = \mathbb{C}^2$

$$= \widehat{\bigoplus_{k > 0}} H_k \otimes (\mathbb{C}^2)$$

$$= \widehat{\bigoplus} H_k \oplus H_k$$

*) $G = Gl_{n+1}\mathbb{C}$

$H = Gl_{1,n}\mathbb{C} = \left\{ A \in Gl_{n+1}\mathbb{C} \;\middle|\; A = \begin{pmatrix} A_{11} & \sim \\ 0 & \sim \\ 0 & \sim \\ \vdots & \vdots \\ 0 & \sim \end{pmatrix} \right\}$

$G/H = \mathbb{C}P^n = \mathbb{C}^{n+1}/\mathbb{C}^* = \{$lineas que pasan por el origen en $\mathbb{C}^{n+1}\}$

$G \circlearrowright \mathbb{C}P^n \qquad A \cdot [x_0 : x_1 : \cdots : x_n] = [Ax_0 : Ax_1 : \cdots : Ax_n]$

Esta acción es transitiva, además el estabilizador de $[e_0] = [1:0:\cdots:0]$ es $G_{[e_0]} = H$

Por lo tanto $\mathbb{C}P^n \cong G \cdot [e_0] \cong G/G_{[e_0]} = G/H$

V_k = pol homogéneos en $\mathbb{C}^{n+1}$ de grado k

$G \curvearrowright V_k$

$$\underbrace{V_k^{hol} \cong \mathrm{Ind}_H^G \mathbb{C}^{hol}}$$

$$\rho : H \longrightarrow GL_1\mathbb{C}$$
$$h \longmapsto \rho h$$
$$(\rho h) z = h_{11}^{-k} z$$

$x \in U \subset M \xrightarrow{f} \mathbb{C}$ es holomorfa si

$U \xrightarrow{h} \Omega \subset \mathbb{C}^k$

$h^{-1} \circ f : \Omega \to \mathbb{C}$ es holomorfa