Lie theory

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Lie Groups, Section 5

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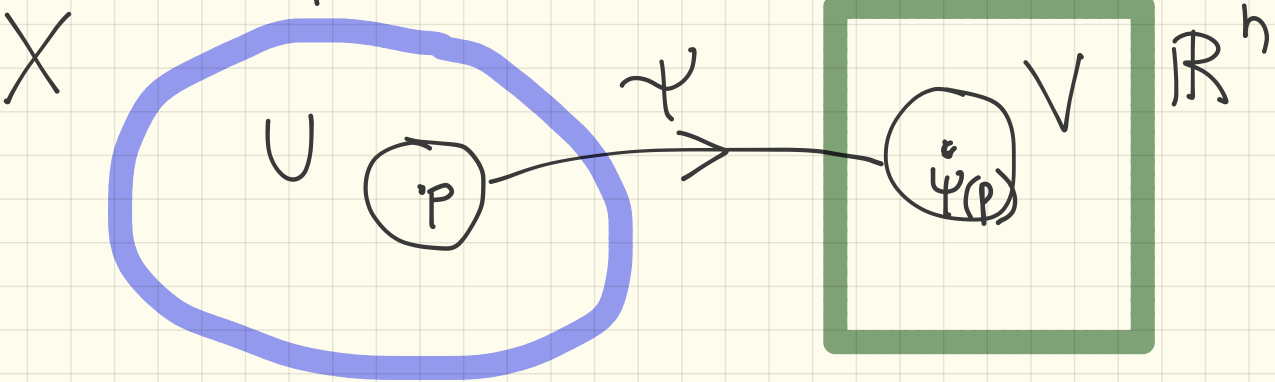
Def (manifold)

A manifold is a topological space X which is locally homeomorphic to some Euclidean space $\mathbb{R}^n$ i.e. each point of X has a neighbourhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$.

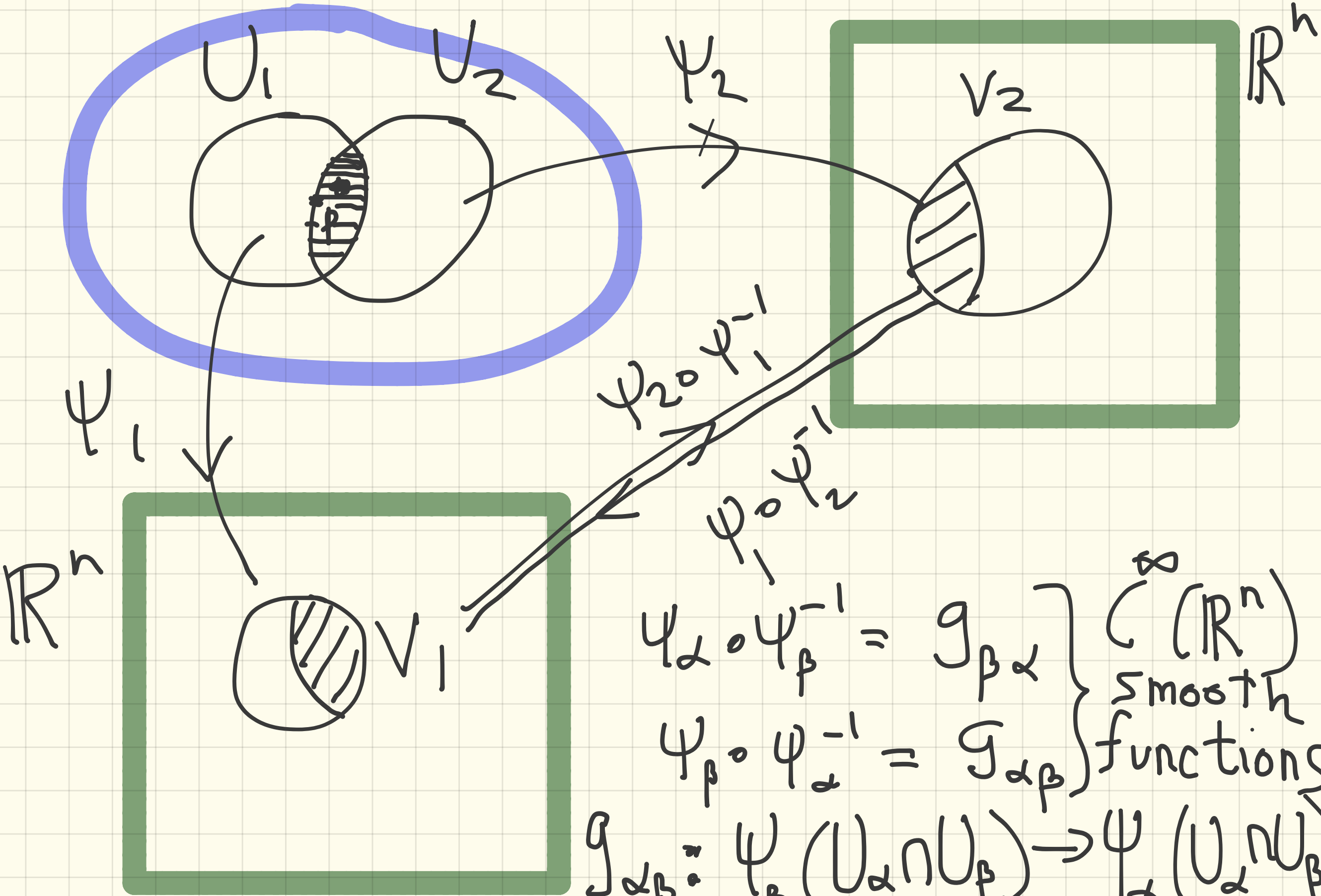


- $\{C_\alpha = (U_\alpha, \psi_\alpha, V_\alpha)\}$ are called charts for the manifold.

Def (smooth mfd)

A smooth mfd X is a pair $(X, \mathcal{A}_X)$ w/ X is a manifold and $\mathcal{A}_X$ is a preferred collection of charts $\psi_\alpha : U_\alpha \to V_\alpha$ which cover all of X and are smoothly related:

- $\mathcal{A}_X$ atlas is maximal
any chart which is smoothly related to all the charts of the atlas belongs to $\mathcal{A}_X$.



$$\left.\begin{aligned} \psi_\alpha \circ \psi_\beta^{-1} &= g_{\beta\alpha} \\ \psi_\beta \circ \psi_\alpha^{-1} &= g_{\alpha\beta} \end{aligned}\right\} C^\infty(\mathbb{R}^n) \text{ smooth functions}$$

$$g_{\alpha\beta} : \psi_\beta(U_\alpha \cap U_\beta) \to \psi_\alpha(U_\alpha \cap U_\beta)$$

Example: $X = S^2$

$$S^2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

covered by six open sets:

$U_1 =$ points w/ $(x>0)$

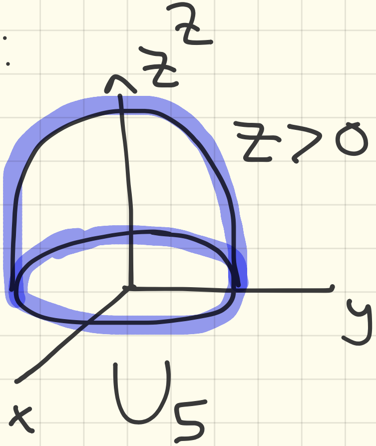
$U_2 =$ " " $(x<0)$

$U_3 =$ " " $(y>0)$

$U_4 =$ " " $(y<0)$

$U_5 =$ " " $(z>0)$

$U_6 =$ " " $(z<0)$



- Charts: $\psi_i : U_i \to V_i$

$\psi_1(x,y,z) = (y,z)$, $\psi_1^{-1}(y,z) = ((1-y^2-z^2), y, z)$

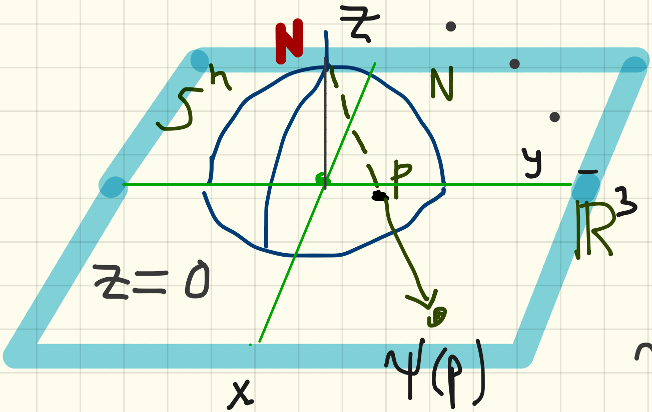
- These charts are smoothly related

$$\psi_{13} = \psi_3 \circ \psi_1^{-1} = \left((1-y^2-z^2)^{1/2}, z\right)$$ Smooth maps

and similarly for the other charts.

- Another chart belonging to the same atlas:

stereographic projection for $N = (0,0,1)$

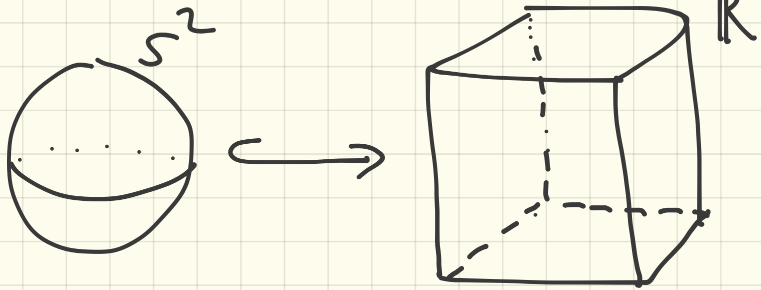


$$S^2$$

$$\psi : U = S^2 - \{N\} \to \mathbb{R}^2$$

$$\psi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

$x_1 = x$; $x_2 = y$; $x_3 = z$



S^2 submfd of $\mathbb{R}^3$

- $O_3 \hookrightarrow \mathbb{R}^9$

embedded naturally in $\mathbb{R}^9$

$$O_3 = \{ A \; ; \; A^t A = 1 \}$$

3×3 matrices defined by 6 equations

- Charts for O_3 (Cayley parametrization)

$$U = \{ A \in O_3 \; ; \; \det(A+1) \neq 0 \}$$

$$\underset{\text{orthog}}{O} = \frac{B+1}{B-1} \; , \quad \underset{\text{skew symm}}{B} = \frac{O-1}{O+1}$$

- O_3 is covered by the open sets

$$\{ gU \} \; ; \quad g \in O_3$$

- **Bijection**

$$\psi : U \to V \cong \mathbb{R}^3$$

w/ $V =$ skew 3×3 matrices

$$\psi(A) = (A-1)(A+1)^{-1}$$ is a skew-symm matrix

- **Charts** are given by

$$\psi_g : gU \to V$$

$$A \mapsto \psi_g(A) = \psi(g^{-1}A)$$

Projective Space

$$\mathbb{P}^{n-1}_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^n)$$

Set of lines through the origin in $\mathbb{R}^n$.

- $(x_1, \dots, x_n) \in \mathbb{P}^{n-1}_{\mathbb{R}}$ s.t. $(x_1, \dots, x_n)$ not all zero (homogeneous coordinates)

  $$(x_1, \dots, x_n) \sim (\lambda x_1, \dots, \lambda x_n) \; ; \; \lambda \neq 0$$

- $U_n \subset \mathbb{P}^{n-1}_{\mathbb{R}}$ part consisting of points w/ $x_n \neq 0 \implies$

- bijection: $\psi_n : U_n \to \mathbb{R}^{n-1}$

  $(x_1, \dots, x_n) \mapsto \psi_n(x_1, \dots, x_n) = \left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)$

- $\mathbb{P}^{n-1}_{\mathbb{R}}$ is covered by n sets $U_1, \dots, U_n$ w/ bijections

  $\psi_i : U_i \to \mathbb{R}^{n-1}$

  define a smooth atlas.

Let X & Y be smooth manifolds

Use charts:

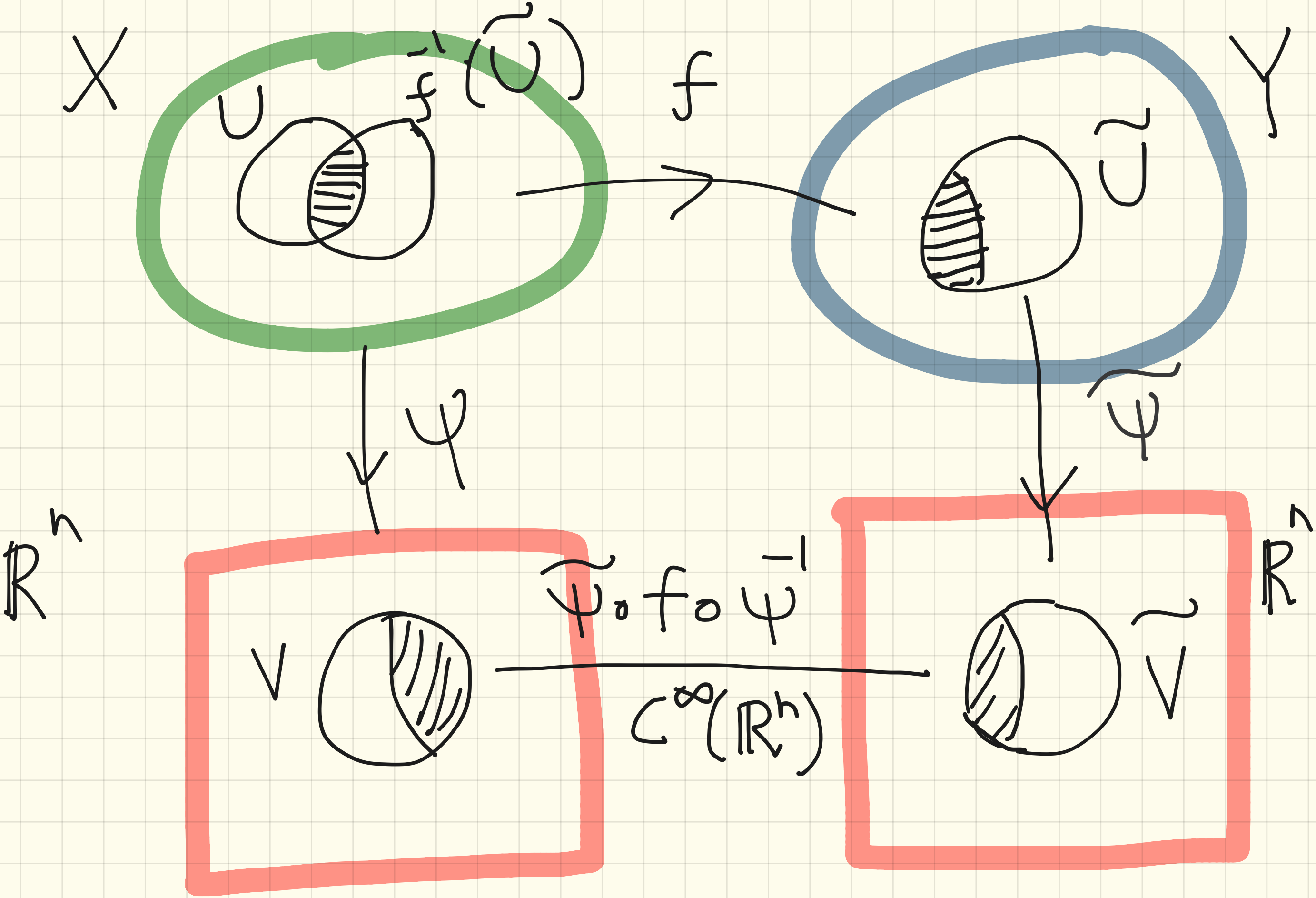
$f : X \to Y$ is smooth

if $\tilde{\psi} \circ f \circ \psi^{-1}$ is smooth map

$$\tilde{\psi} \circ f \circ \psi^{-1} : \psi(U \cap f^{-1}(\tilde{U})) \to \tilde{V}$$

w/ $\psi : U \to V$ $\quad$ $\tilde{\psi} : \tilde{U} \to \tilde{V}$

are charts of X & Y.



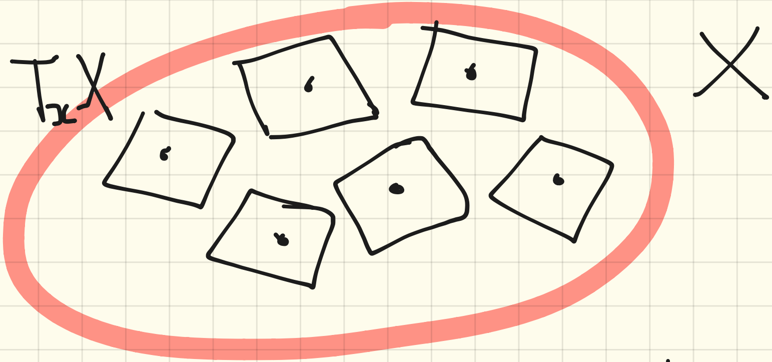
X
U
$f^{-1}(\tilde{U})$
f
Y
$\tilde{U}$
ψ
$\tilde{\psi}$
$\mathbb{R}^n$
V
$\tilde{\psi} \circ f \circ \psi^{-1}$
$C^\infty(\mathbb{R}^n)$
$\mathbb{R}^n$
$\tilde{V}$

- **Def (Lie group)**
  A Lie group is a smooth mfd G together w/ a smooth map $\mu : G \times G \to G$ which makes it a group.

- Any closed subgroup of $GL_n(\mathbb{R})$ is a Lie group. (see Adams)

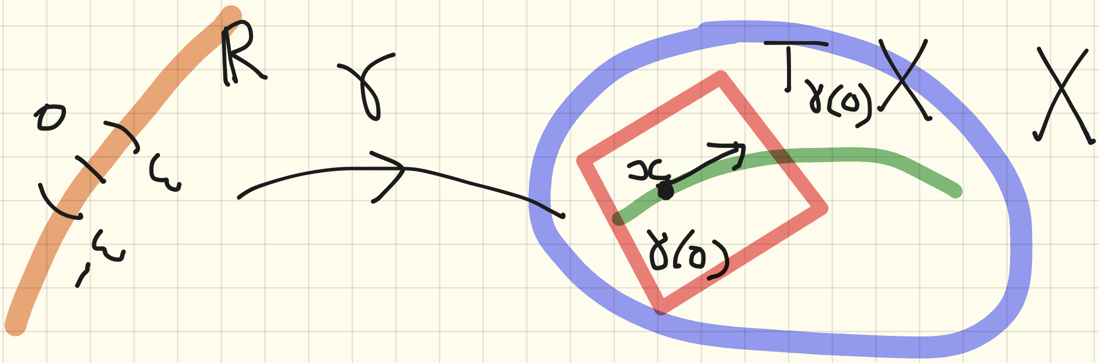
- Implicit function theorem to solve $xy = 1$
  $\Rightarrow$ in any Lie group, the map $G \to G$, $x \mapsto x^{-1}$ is a smooth map.

Tangent Spaces



- A smooth n-dimensional mfd has a **tangent space** @ each point $x \in X$.
- T_xX is an n-dimensional real vector space.
- If $\underbrace{X \subset \mathbb{R}^N}_{\text{submfd}}$. $\underbrace{T_xX \subseteq \mathbb{R}^N}_{\text{vect subspace}}$

- Consider all smooth curves



$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow X$$
$$0 \mapsto \gamma(0) = x$$

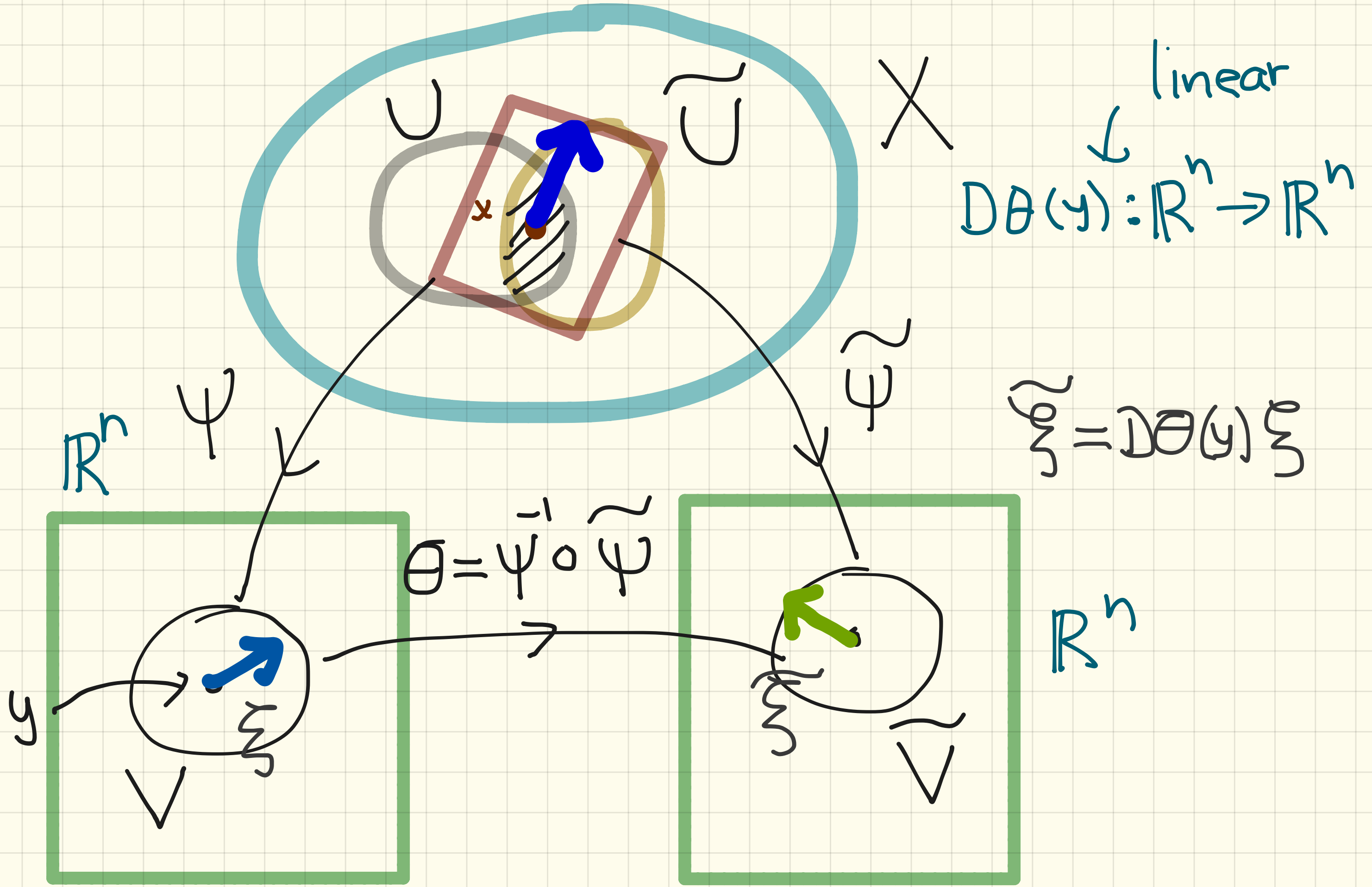
- $T_x X$ = set of all velocity vectors @ $\gamma(0)$
  $\gamma'(0) \in \mathbb{R}^N$

- Define $T_x X$ without invoking the ambient space $\mathbb{R}^N$.

$(x, \psi, \xi) \in T_x X$; $\psi : U \to V$ is a chart s.t. $x \in U$, ξ vector in $\mathbb{R}^n$ (representative of an element of $T_x X$ wrt the chart ψ.

- The triple $(x, \psi, \xi) \sim (x, \tilde{\psi}, \tilde{\xi})$ iff

$\tilde{\xi} = D\theta(y)\xi$; $\theta = \tilde{\psi} \circ \psi^{-1}$ in a neighbourhood of y.



X
U
$\tilde{U}$
x
linear
$D\theta(y): \mathbb{R}^n \to \mathbb{R}^n$
ψ
$\tilde{\psi}$
$\tilde{\xi} = D\theta(y)\xi$
$\mathbb{R}^n$
$\theta = \psi^{-1} \circ \tilde{\psi}$
y
ξ
V
$\tilde{\xi}$
$\tilde{V}$
$\mathbb{R}^n$

Example: $G = O_n \underset{\text{mfd}}{\subset} GL_n(\mathbb{R})$

$T_1 G = S$ w/ $\dim S = \frac{1}{2} n(n-1)$ skew matrices

$T_g G = gS = Sg$

Proof: For any skew .natrix A

$e^{tA} \in O_n \Rightarrow \gamma(t) = g e^{tA}$

w/ $\gamma : \mathbb{R} \to G$ is a path such that

$\gamma(0) = g$; $\gamma'(0) = gA$

$\gamma : (-\varepsilon, \varepsilon) \to G$ is a path such that

$$\gamma(0) = g \Rightarrow \frac{d}{dt}(\gamma^t \cdot \gamma) = \frac{d}{dt}(\mathbb{1})$$

$$\gamma'(0)^t g + g^t \gamma'(0) = 0 \Rightarrow (g^t \gamma'(0))^t = -g^t \gamma'(0)$$

$$\Rightarrow g^t \gamma'(0) \text{ is skew, } g^t g A = A \in T_1 G.$$

$\square$

Exercise: $G = U_n \Rightarrow T_1 G$ is a n^2 dim. real vector space of skew hermitean matrices.

Notation

(i) Smooth map $f : X \to Y \rightsquigarrow$ a linear map $\forall\ x \in X$

$$T_x X \xrightarrow{Df(x)} T_{f(x)} Y$$

(ii) G = Lie group, $g \in G$, $\exists$ a smooth map $L_g : G \to G$, $L_g(x) = gx$. This induces

left translation

$$DL_g(x) : T_x G \to T_{gx} G$$

isomorphism

$$\xi \longmapsto DL_g(x)(\xi) = g \cdot \xi .$$

- The corresponding isomorphism

$$T_x G \to T_{xg} G \quad \text{right translation}$$
$$\xi \longmapsto \xi g$$

One-parameter subgroups & the exponential map

- homomorphism: $f : \mathbb{R} \to GL_n(\mathbb{R})$

is a one parameter subgroup

$$f(t) = e^{tA}$$

w/ $A = f'(0)$.

- Take

$$f'(t) = \lim_{h\to 0}\left[\frac{f(t+h)-f(t)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(h)-1}{h}\right]$$

$$\boxed{f'(t) = A f(t)}$$

- The unique solution of this differential equation w/ $f(0) = 1$ is

$$f(t) = e^{tA}$$

- **Exponential map**:

$$\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$$

is bijective in a neighbourhood of zero, its **inverse** being the smooth map

$$g \mapsto \log g$$

defined by:

$$\log(1-A) = \sum_k \frac{A^k}{k}$$

Theorem

In any Lie group G, $\exists$ a 1-1 correspondence between T_1G and the homomorphisms

$$f : \mathbb{R} \to G$$

Proof:

$\Rightarrow$) Same argument as for $GL_n(\mathbb{R})$

$$f \rightsquigarrow f'(0) \in T_1 G$$

$\Leftarrow$) $A \in T_1 G$ defines a tangent vector field ξ_A on G by

$$\xi_A(g) = Ag$$

- ξ_A has a unique solution curve w/ $$f(0) = 1$$
- Theory of ODEs gives a solution
$$f : (-\varepsilon, \varepsilon) \longrightarrow G$$
$$t \mapsto f(t+u)$$
$$t \mapsto f(t) f(u)$$
are solution curves of ξ_A w/
$$f(u) \quad \text{for} \quad t = 0$$

For any $t \in \mathbb{R}$

$$\left[f\left(\frac{t}{n}\right)\right]^n$$

is defined for all sufficiently large n & is independent of n because

$$f\left(\frac{t}{n}\right)^n = f\left(\frac{t}{nm}\right)^{nm} = f\left(\frac{t}{m}\right)^m$$

$\Rightarrow$ define $f(t) = f\left(\frac{t}{n}\right)^n$ for any large n.

- We have therefore the map

$$\exp : T_1 G \longrightarrow G$$

whose derivative @ 0 is the identity.

- In general **exp** is neither 1-1 nor onto, but by the **inverse function theorem** $\exists$ a smooth inverse map 'log' defined in the neighbourhood of $1 \in G$.

Examples:

(i) If $G = SL_n(\mathbb{R})$ then $T_1 G$ are the $n \times n$ matrices w/ trace 0 because

$$\det(e^{tA}) = e^{tr(A)}$$

(ii) If $G = SU_2$ then $T_1 G$ is the skew-hermitean matrices w/ trace 0. i.e. pure vector quaternions $\mathbb{R}^3$.

If $u \in \mathbb{R}^3$ is a unit vector $\Rightarrow$

$$u^2 = -1.$$

- $e^{tu} = \cos t + u \sin t$

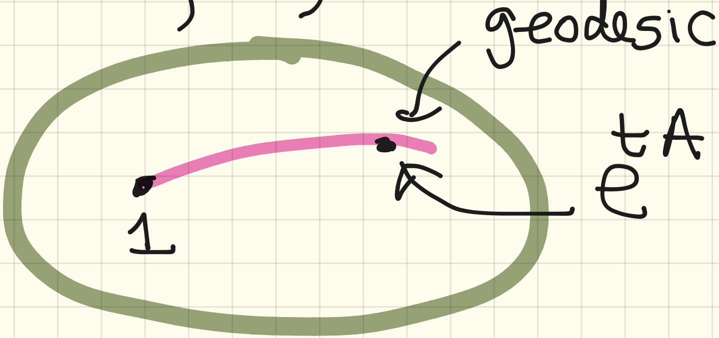
one-parameter subgroup of rotations about u. exp is surjective.

- Remark: exp is surjective in any compact group G.

metric

If (G, h) Riemannian mfd

G



in a complete Riemannian mfd any 2 points can be joined by a geodesic.

Lie's theorems

- In a Lie group G w/ $T_1 G = \mathfrak{g}$

$$\log : U \to \mathfrak{g}$$

is a canonical chart defined on $1 \in U$

- Composition law: $G \times G \to G$ in this chart:

$$C(A,B) = \log(\exp(A) \cdot \exp(B))$$

in terms of $A, B \in \mathfrak{g}$

- Taylor series @ $A=B=0$

$$C(A,B) = A+B+\frac{1}{2}b(A,B) + O(\text{order} \geq 3) \quad (*)$$

w/ $C(A,0)=A$; $C(0,B)=B$

- b is a bilinear map

$$b : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

b is skew since

$$C(-B,-A) = -C(A,B)$$

w/ $b(-B,-A) = -b(A,B)$

Basic miracle of Lie's theory:

- Infinite series $(\star)$ can be expressed enterely in terms of $b(A,B)$.
- The series converges in a neighbourhood of the origin.

eg. third order terms:

$$\frac{1}{12}\, b(A, b(A,B)) + \frac{1}{12}\, b(B, b(B,A))$$

- Complete series $(\star)$ is called the Campbell–Baker–Hausdorff series

- In a matrix group:

$$b(A,B) = [A,B] = AB - BA$$

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

- $[A,B] = -[B,A]$ skew symmetric

- $[[A,B],C] + [[B,C],A] + [C,[A,B]] = 0$

  Jacobi identity.

$\Rightarrow$ $\mathfrak{g}$ is a Lie algebra

- **Example:** $G = SO_3 \Rightarrow \mathfrak{g} =$ 3×3 real skew matrices $= \mathbb{R}^3$

Lie bracket

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

is the "croos product" $\times$

- $a \times b = -b \times a$ skew symmetric
- $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$

Jacobi identity

w/ $(a \times b) \times c = \langle a, c \rangle b - \langle b, c \rangle a$

- **Theorem (Lie)**

The functor taking G to $T_1 G = \mathfrak{g}$ is an equivalence of categories between the category of connected & simply connected Lie groups and the category of Lie algebras. $\blacksquare$

- **Implications:** Every Lie algebra arises from a simply connected Lie group G, and that G is determined up to isomorphism.
- Furthermore: $G_1 \to G_2$ homomorphisms are in 1-1 correspondence $T_1 G_1 \to T_1 G_2$ homomorphisms

Sketch of the proof

(i) Groups locally isomorphic have the same Lie algebra.

(ii) $\theta : T_1 G_1 \to T_1 G_2$ homomorphism of Lie algebras

$\exists$ at most

$f : G_1 \to G_2$ group homomorphism

which induces θ.

i.e. $\theta \rightsquigarrow f$ determined by its restriction to $1 \in U$

$\Longrightarrow$

$$\begin{tikzcd} T_1 G_1 \arrow[r, "\theta"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\cong"] & T_1 G_2 \arrow[d, "\exp"] \\ G_1 \arrow[r, "f"] & G_2 \end{tikzcd}$$

$$f(\exp \xi) = \exp \theta(\xi)$$

- $t \mapsto f(\exp \xi)$
  $t \mapsto \exp \theta(\xi)$ } 1-parametric subgroups of G_2 w/ the same derivative @ $t=0$

(iii) If $\mathfrak{h}$ is a sub-Lie algebra of $\mathfrak{g} = T_1 G$ $\exists$ a Lie group H w/ $T_1 H = \mathfrak{h}$ & a homomorphism $H \to G$ inducing the inclusion: $\mathfrak{h} \hookrightarrow T_1 G$ i.e.

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, hook, "i"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\
H \arrow[r, "\text{homomorphism}"'] & G
\end{tikzcd}$$

Proof of Lie's theorem

Problem: Construct a homomorphism of Lie groups

$$f : G_1 \to G_2$$

when one is given a homomorphism

$$\theta : \mathfrak{g}_1 \to \mathfrak{g}_2$$

of Lie algebras.

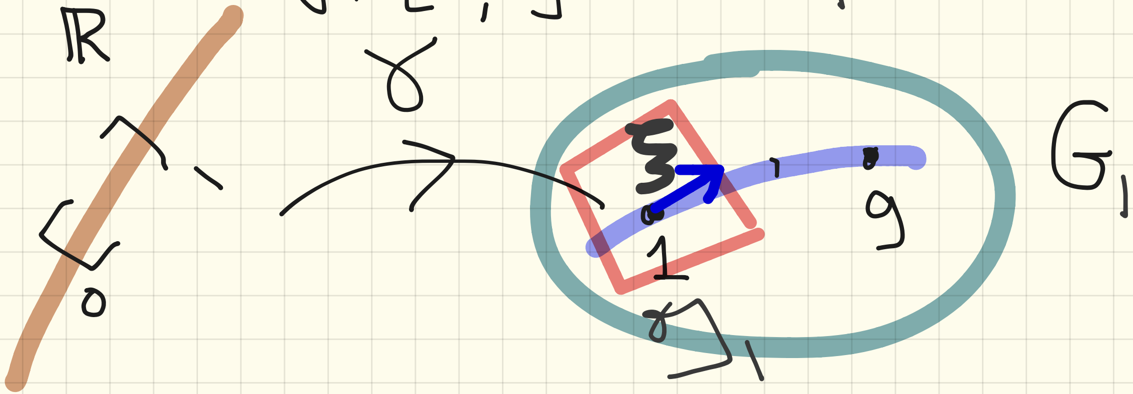
- In a neighbourhood of the identity f is defined as

$$\boxed{f(\exp \xi) = \exp \Theta(\xi)}$$

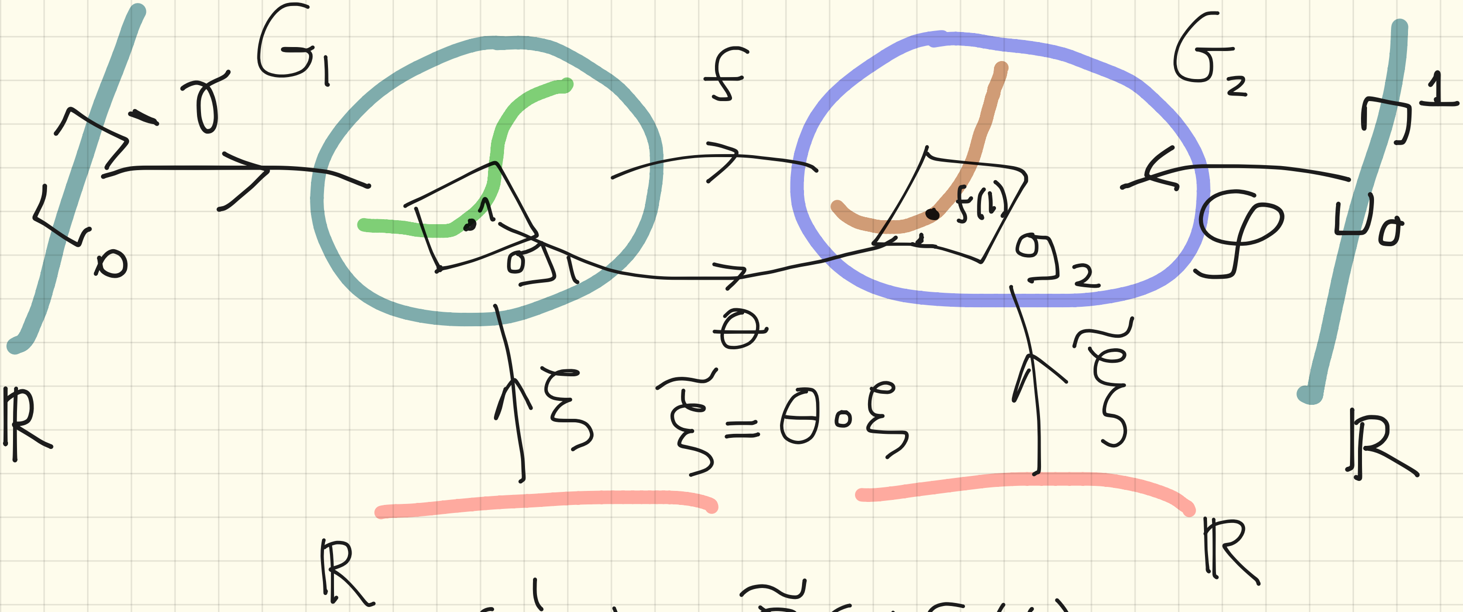
- One way to prove the theorem:
Show that this is a homomorphism (where it is defined) by constructing the CBH series explicitly & proving that it converges (see Serre).
- Extend f to the whole group using that G, $\pi_0(G) = 0$ & $\pi_1(G) = 0$.

Different route:

- To define $f(g)$ choose a smooth path
$$\gamma : [0,1] \longrightarrow G_1$$



- Consider the path
$$t \longmapsto \frac{\partial \gamma}{\partial t}(t)\, \gamma^{-1}(t) = \xi(t)$$
in $\mathfrak{g}_1$



$$\varphi'(t) = \tilde{\xi}(t)\varphi(t)$$

w/ $\quad \varphi(0) = 1$

& define: $\quad f(g) = \varphi(1)$

- Define $f(g) = \varphi(1)$.
- The main point is to show that $\varphi(1)$ does not depend on the choice of the path γ from 1 to g.
- Consider $\pi_1(G) = 0$, take two paths in the family

$$\{t \mapsto \gamma_s(t)\}_{0 \le s \le 1}$$

all from 1 to g.

- Let

$$\xi(t,s) = \frac{\partial}{\partial t}\gamma_s(t)\cdot\gamma_s^{-1}(t) \in \mathfrak{g}_1$$

$$\eta(t,s) = \frac{\partial}{\partial s}\gamma_s(t)\cdot\gamma_s^{-1}(t) \in \mathfrak{g}_1$$

and calculate the Maurer-Cartan equation:

$$\frac{\partial \xi}{\partial s} - \frac{\partial \eta}{\partial t} = [\eta,\xi] \qquad (\star)$$

- Define $\tilde{\xi} = \Theta\circ\xi$; $\tilde{\eta} = \Theta\circ\eta \Rightarrow$

$$\frac{\partial \tilde{\xi}}{\partial s} - \frac{\partial \tilde{\eta}}{\partial t} = [\tilde{\eta},\tilde{\xi}] \qquad (\star\star)$$

- This is the compatibility condition which enables to solve equations #

$$\boxed{\frac{\partial \varphi}{\partial t} = \tilde{\xi}\,\varphi} \; ; \; \boxed{\frac{\partial \varphi}{\partial s} = \tilde{\eta}\,\varphi} \qquad (\star\star\star)$$

to obtain:

$$\varphi : [0,1]\times[0,1] \to G_2 \; ; \; \varphi = \varphi(s,t)$$

- Eqns: $(\star)$ & $(\star\star)$ express the fact that the Lie-algebra-valued 1-forms

$$A = \xi\, dt + \eta\, dS$$

$$\tilde{A} = \tilde{\xi}\, dt + \tilde{\eta}\, dS$$

are flat connections on $\mathbb{R}^2$.

$\Rightarrow$) Starting from Eqs. (★★★) $\Rightarrow$

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\varphi = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi \quad \Rightarrow$$

$$\frac{\partial}{\partial s}\frac{\partial \varphi}{\partial t} = \frac{\partial \tilde{\xi}}{\partial s}\cdot\varphi + \tilde{\xi}\frac{\partial \varphi}{\partial s}$$

$$\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial s} = \frac{\partial \tilde{\eta}}{\partial t}\cdot\varphi + \tilde{\eta}\frac{\partial \varphi}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \tilde{\xi}}{\partial s}\cdot\varphi + \tilde{\xi}\frac{\partial \varphi}{\partial s} = \frac{\partial \tilde{\eta}}{\partial t}\cdot\varphi + \tilde{\eta}\cdot\frac{\partial \varphi}{\partial t}$$

$$\left[\frac{\partial \tilde{\xi}}{\partial s} - \frac{\partial \tilde{\eta}}{\partial t}\right]\varphi = \tilde{\eta}\cdot\frac{\partial}{\partial t}\varphi - \tilde{\xi}\frac{\partial \varphi}{\partial s}$$

$$= \left(\tilde{\eta}\frac{\partial}{\partial t} - \tilde{\xi}\frac{\partial}{\partial s}\right)\varphi$$

$\Rightarrow$ $$\frac{\partial\tilde{\xi}}{\partial s} - \frac{\partial\tilde{\eta}}{\partial t} = [\tilde{\eta}, \tilde{\xi}]$$

$\Leftarrow$) If $\frac{\partial\tilde{\xi}}{\partial s} - \frac{\partial\tilde{\eta}}{\partial t} = [\tilde{\eta}, \tilde{\xi}]$ holds $\Rightarrow$

define $\varphi(t,0)$ by integrating

$$\frac{\partial\varphi}{\partial t} = \tilde{\xi}\varphi(t)$$

along line $s=0$ $\Rightarrow$ define

$\varphi(t,s)$ by integrating

$$\frac{\partial\varphi}{\partial s} = \tilde{\xi}\varphi$$

holding $t =$ constant.

$\Rightarrow$ (★★) $\Rightarrow \dfrac{\partial}{\partial s}\left\{\dfrac{\partial \varphi}{\partial t} - \tilde{\xi}\varphi\right\} = 0$

and
$$\frac{\partial}{\partial t}\left\{\frac{\partial \varphi}{\partial s} - \tilde{\eta}\varphi\right\} = 0$$

$\Rightarrow \varphi$ satisfies both equations (★★★) $\blacksquare$

- Eqns (☆) & (☆☆) can be written as

$$dA = \frac{1}{2}[A, A]$$

$$d\tilde{A} = \frac{1}{2}[\tilde{A}, \tilde{A}]$$

- Proof.

$$d(\xi dt + \eta ds) = \frac{1}{2}[\xi dt + \eta ds, \xi dt + \eta ds]$$

$$\frac{\partial \xi}{\partial s} ds \wedge dt + \frac{\partial \eta}{\partial t} dt \wedge ds$$

$$= \frac{1}{2}[\xi, \eta] dt \wedge ds + \frac{1}{2}[\eta, \xi] ds \wedge dt$$

$$= \frac{1}{2}[\eta, \xi] ds \wedge dt + \frac{1}{2}[\eta, \xi] ds \wedge dt$$

- $\left(\frac{\partial \xi}{\partial s} - \frac{\partial \eta}{\partial t}\right) ds \wedge dt = [\eta, \xi]\, ds \wedge dt \Rightarrow$

$$\frac{\partial \xi}{\partial s} - \frac{\partial \eta}{\partial t} = [\eta, \xi]$$

- $\eta, \bar{\eta}$ vanish w/ $t=1$ by definition $\Rightarrow$ $\frac{\partial \varphi}{\partial s} = 0$ w/ $t=1$ & $\varphi(1,s)$ is independent of s.
- Most difficult part of the Lie's thm is the proof that any (finite dimensional) Lie algebra arises from a Lie group.

- Use Ado's theorem

$$\mathfrak{g} \underset{\text{iso}}{\cong} \text{subalgebra of } \mathrm{Lie}(M_n(\mathbb{R}))$$

- Consider all smooth maps

$$\xi : [0,1] \to \mathfrak{g} \subset M_n(\mathbb{R})$$

s.t. $\xi(0) = \xi(1) = 0$

$\xi'(0) = \xi'(1) = 0$

- For each ξ we solve the ode

$$\varphi_\xi'(t) = \xi(t)\varphi_\xi(t)$$

w/ $\varphi_\xi(0) = \mathbb{1}$

- find $\varphi_\xi : [0,1] \longrightarrow GL_n(\mathbb{R})$
- elements $\varphi_\xi(1) \subset GL_n(\mathbb{R})$ is a subgroup for

$$\varphi_\eta(1)\, \varphi_\xi(1) = \varphi_{\eta * \xi}(1) \qquad (1)$$

w/ $\eta * \xi : [0,1] \to \mathfrak{g}$ is the concatenation of ξ and η i.e.

$$(\eta * \xi)(t) = \begin{cases} 2\xi(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ 2\eta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

- To prove (1) we observe that, if

$$\varphi(t) = \begin{cases} \varphi_\xi(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ \varphi_\eta(2t-1)\,\varphi_\xi(1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

$\Rightarrow$ φ satisfies: $\varphi' = (\eta * \xi)\,\varphi$.

- Subgroup of $GL_n(\mathbb{R})$, $\varphi_\xi(1)$ is not closed.

- Instead consider the vector space $\mathcal{F}$ of maps ξ & consider the equivalence relation

- $\xi_0 \sim \xi_1 \Leftrightarrow \varphi_{\xi_0}(1) = \varphi_{\xi_1}(1)$
- Quotient space $\mathcal{P}/\sim$ = topological group under the operation of concatenation
- $\mathcal{P}/\sim$ is the desired group
- $\mathcal{P}/\sim$ is locally homeomorphic to $\mathfrak{J}$.
- If $\xi \in \mathcal{P}$ is small $\Rightarrow \hat{\varphi}_\xi = \log \varphi_\xi$ is a well defined path in $M_n(\mathbb{R})$.

- This is contained in $\mathfrak{g}$ $\Rightarrow$
  $\mathcal{P}/\sim$ is locally the same as the space of smooth paths.
- Smooth paths:
  $\hat{\varphi} : [0,1] \to \mathfrak{g}$
  w/ $\hat{\varphi}(0) = 1$ modulo
  $\hat{\varphi}_0 \sim \hat{\varphi}_1 \Leftrightarrow \hat{\varphi}_0(1) = \hat{\varphi}_1(1) \Rightarrow$
  locally: $\mathcal{P}/\sim \; = \mathfrak{g}$.
- The composition law in $\mathcal{P}/\sim$ is smooth.

- Remains to prove that $\hat{\varphi}_\xi \subset \mathfrak{g}$
- Velocity $\hat{\varphi}_\xi'(t)$ is related to

$$\varphi_\xi'(t) \quad \text{and} \quad \varphi_\xi'(t)\,\varphi_\xi^{-1}(t) \in \mathfrak{g}$$

- For any Lie group

$$\delta(e^A)\,e^{-A} = F(\mathrm{ad}\,A)\,\delta A \qquad (2)$$

w/ $F : \mathrm{End}(M_n(\mathbb{R})) \longrightarrow \mathrm{End}(M_n(\mathbb{R}))$

$$F(x) := \frac{e^x - 1}{x} = \sum_{k \geq 0} \frac{x^k}{(k+1)!}$$

- $\operatorname{ad} A \in \operatorname{End}(M_n(\mathbb{R}))$ is given by

$$\operatorname{ad} A(B) = [A, B]$$

- Formula (2) shows that

$$A(t) = \hat{\varphi}_{\xi}(t)$$

satisfies the differential equation

$$F(\operatorname{ad} A)A' = \xi(t)$$

for $A : [0,1] \to \mathfrak{g}$.

this completes the proof $\blacksquare$

Derivation of (2).

Combine 2 results:

$$\frac{d}{dt} e^{A} = \int_0^1 e^{sA} \frac{dA}{dt} e^{(1-s)A} \, ds$$

for any function $A : \mathbb{R} \to M_n(\mathbb{R})$

$$e^{\mathrm{ad}A}(B) = e^{A} B e^{-A}$$

- How to avoid invoking Ado's thm.
- It was used to define the equivalence relation

$$\xi_0 \sim \xi_1 \iff \varphi_{\xi_0}(1) = \varphi_{\xi_1}(1)$$

on $\mathfrak{P}$.

- Replace by

$$\xi_0 \sim \xi_1 \iff \xi_0 \text{ \& } \xi_1 \text{ are joined by a path } \xi_s \text{ in } \mathfrak{P}$$

such that

$$\frac{\partial \xi}{\partial s} - \frac{\partial \eta}{\partial t} = [\eta, \xi]$$

is satisfied for some η.

- $\mathfrak{P}/\sim$ = topological group

- To prove that it is locally like $\mathfrak{G}$

$$\xi \in \mathfrak{P} \longleftrightarrow \hat{\varphi}_\xi \text{ path in } \mathfrak{G}$$

by solving:

$$F(\mathrm{ad}\,A)A' = \xi(t)$$

- One must check that

$$\xi_0 \sim \xi_1 \Longleftrightarrow \hat{\varphi}_{\xi_0}(1) = \hat{\varphi}_{\xi_1}(1).$$