Compact Groups and Integration

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Preliminaries

Topological Vector Space

Suppose $\mathcal{T}$ is a topology on a vector space X such that

- every point in X is a *closed set*[a] and
- the vector space operations are *continuous* with respect to $\mathcal{T}$.

Under these conditions, $\mathcal{T}$ is called a vector topology on X, and X is said to be a topological vector space.

[a] A more precise way to establish this: the set $\{x\}$ which has x as its unique element is a closed set.

Preliminaries

Let's denote

$$T_a(x) = x + a\,, \quad M_\lambda(x) = \lambda x\,. \tag{1}$$

where $x, a \in X$ and λ is a scalar different to zero.

Preposition 1

$T_a(x)$ and $M_\lambda(x)$ are homeomorphisms of X onto X.

Every vector topology $\mathcal{T}$ is *translation-invariant*. A set $E \subset X$ is open if and only if each of its translates $a + E$ is open

- X is *locally convex* if there is a local bases $\mathfrak{B}$ [1] whose members are convex
- X is *locally bounded* if 0 has a bounded neighborhood

[1] In the context of vector spaces, a local bases will mean a local base at 0

Preliminaries

In fact, one more property related to TVS:

- X is *normable* if and only if X is locally convex and locally bounded.

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Finally

Convex Sets

A set E in a vector space X is said to be convex if it has the following geometric property: whenever $x \in E$ and $y \in E$, and $0 < t < 1$, the point

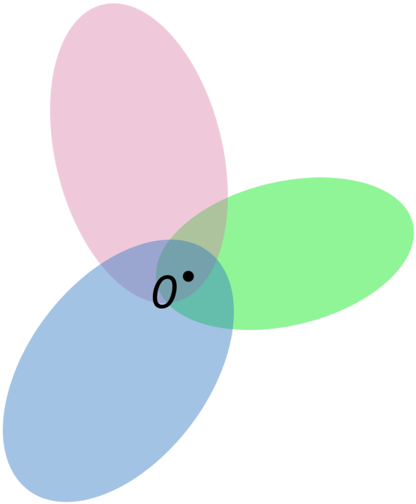
$$z_t = (1 - t)x + ty$$

also lies in E

- Every subspace of X is convex,
- also, if E is convex, then is each of its translates

$$E + a = \{x + a |\ x \in X\}$$

Preliminaries



Previous information can be revised, in a more complete way, e.g. [5, 6]

Integration over manifolds

Definition

Let M be a connected manifold with a maximal atlas $\{(U, \varphi)\}$. M is orientable if, for any coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) with $U_i \cap U_j \neq \emptyset$ then $J > 0$.

There exists an n-form ω which vanishes nowhere. It is called a volume element.

We say that

$$\omega \sim \omega' , \quad \omega = h\omega' \tag{2}$$

Take an n-form

$$\omega = h(p) dx^1 \wedge \cdots \wedge dx^n \tag{3}$$

Integration on manifolds

In turns out that

$$\omega = h(p)dx^1 \wedge \cdots \wedge dx^n = h(p)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^1 \wedge \cdots \wedge dy^n \tag{4}$$

with

$$J = \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0\,, \tag{5}$$

we say that M is orientable and ω may be extended throughout M.

Integration on manifolds

If $f : M \to \mathbb{R}$ over an orientable manifold, then

$$\int_{U_i} f\omega \equiv \int_{\phi(U_i)} f(\phi_i^{-1}(x))h(\phi_i^{-1}(x))dx^1 \wedge \cdots \wedge dx^n \,. \tag{6}$$

The integration of f on M is given by

$$\int_M f\omega \equiv \sum_i \int_{U_i} f_i\omega \,. \tag{7}$$

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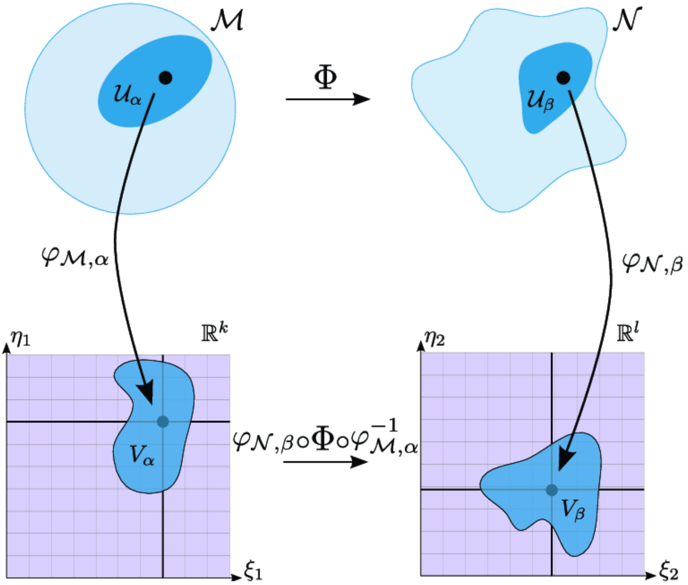
Hold your horses!

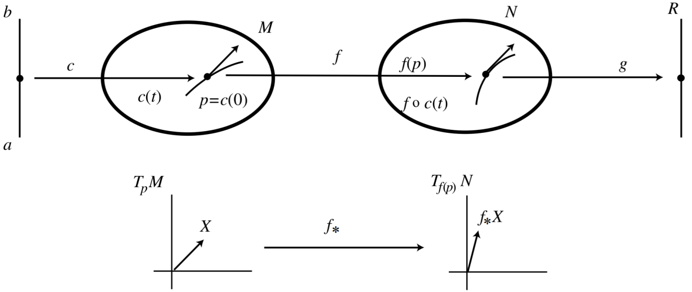
Tangent Space



Tangent Space

A smooth map $f : M \to N$ naturally induces a map

$$f_* = T_p M \to T_{f(p)} N \tag{8}$$



$$(f_* V)[g] \equiv V[g \circ f]\,. \tag{9}$$

(Compact) Lie Groups

Left-invariant Vector Field

Let a and g be elements of a Lie Group G. The *rith-translation* $R_a : G \to G$ and the *left-translation* $L_a : G \to G$ of g by a are defined by

$$R_a g = ga \tag{10}$$

$$L_a g = ag\,, \tag{11}$$

differentiable invertible map from G onto G.

Definition

Let X be a vector field on a Lie group. X is said to be left-invariant vector field if

$$L_{a*}X|_g = X|_{ag}\,.$$

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$$L_{a*}X|_g = X^\nu(ag)\frac{\partial}{\partial x^\nu}\bigg|_{ag} \tag{12}$$

(Compact) Lie Groups

A vector $V \in T_e G$ defines a unique left-invariant vector field $X(V)$ throughout G by

$$X(V)|_g = L_{g_*} V \,, \quad g \in G \,. \tag{13}$$

[2]One can show that

$$L_{ab} g = (L_a \circ L_b) g \,, \quad \text{using } (11)$$

and

$$((g \circ f)_* X_p)[h] = ((g_* \circ f_*) X_p)[h] \,, \quad \text{using } (9).$$

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Indeed[2] ,

$$X(V)|_{ag} = L_{(ag)*} V = (L_a \circ L_g) * V = L_{a*} X(V)|_g \,.$$

Conversely,

$$X(V)|_{ae} = X(V)|_a = L_{a*} X(V)|_e = L_{a*} V \,,$$

a left-invariant vector field defines a unique vector $V = X|_e$.[3]

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Compact Group Integration

From hear see, e.g. [7, 1]

Haar Measure

It turns out to be a theorem that every compact Lie group has a unique invariant measure, up to scalar multiplication.

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Haar Measure

It turns out to be a theorem that every compact Lie group has a unique invariant measure, up to scalar multiplication.

$$\int_G 1dg = 1\,. \tag{14}$$

Compact Group Integration

If

- G is a compact group,
- V is a topological vector space,
- $f : G \to V$,

then

$$\int_G f(g)dg = \int_G f \in V\,. \tag{15}$$

More precisely,

$$\int_G : C(G;V) \to V \tag{16}$$

where $C(G;V) = \{f : G \to V\}$, so that

(i) $\int_G f(g)dg = v$ if $f(g) = v$ for all g,

(ii) $\int_G f(hg)dg = \int_G = f(gh)dg = \int_G f(g)dg$ for any $h \in G$, and

(iii) if f takes its values in a convex subset C of V, then $\int_G f \in C\,.$

Compact Group Integration

Suppose that ω is an n-form on N where $n = \dim N = \dim M$ and $\Phi : M \to N$, then

$$\int_M \Phi^* \omega = \int_N \omega \,. \tag{17}$$

If ω does not have compact support,

$$\int_M \Phi^* f \Phi^* \omega = \int_N f \omega \,. \tag{18}$$

Now, let us go to be interested in $M = N = G$ and $\Phi = L_a$,

$$\int_G (r(a) f) L_a^* \omega = \int_G f \omega \,, \tag{19}$$

So, is ω such that

$$L_a^* \omega = \omega \tag{20}$$

for all $a \in G$, then $\int f \omega$ would be the invariant integral.

Compact Group Integration

If we realise this with the simple example $G = \mathbb{R}^+$, then

$$\int_{\mathbb{R}^+} \Phi^*(k(x)dx) = \int_{\mathbb{R}^+} (k \circ \Phi)(x)\Phi^*(dx) \tag{21}$$

$$= \int_{\mathbb{R}^+} (k \circ L_a)(x)L_a^*(dx) \tag{22}$$

$$= \int_{\mathbb{R}^+} (k(b^{-1}x))b^{-1}dx\,. \tag{23}$$

Thus

$$\omega = x^{-1}dx$$

is the left-invariant form, and

$$\int_G f = \int_{\mathbb{R}^+} f(x)x^{-1}dx \tag{24}$$

Compact Group Integration

Theorem

Weyl Integral Formula

For all continuous function f on G, we have

$$\int_G f = k \int_{T\times G/T} J(t) f(gtg^{-1}) d[g] dt \tag{25}$$

where dg, dt and $d[g]$ are the normalised, left-invariant volume forms on G, T, and G/T, respectively and k is related to the order of the Weyl group.

We need some *propositions* to prove this theorem which we will not prove it to this talk.

Compact Group Integration

Firstly,

Proposition 2

Let X, Y be closed, oriented manifolds and let $f : X \to Y$ be a smooth map, and suppose f has *mapping degree* k. Then for every n-form ω on Y, we have

$$\int_X f^*(\omega) = k \int_Y \omega \,. \tag{26}$$

Secondly, (**Theorem**) if G is a matrix Lie group and H is a closed subgroup of G, then G/H can be given the structure of smooth manifold

- $Q : G \to G/H$ is smooth,
- the differential of Q at the identity maps $T_I(G)$ onto $T_{[I]}(G/H)$ with *kernel* $\mathcal{L}(H)$, and
- the left action of G on G/H is smooth.

Compact Group Integration

Proposition 3

Suppose there exists an inner product on $\mathcal{L}(G)$ that is invariant under the *adjoint action* of H and let η $(\mathcal{L}(H)^{\perp})$ denote the orthogonal complement of $\mathcal{L}(H)$ with respect to this inner product. Then we may identify the tangent space at each point $[g]$ of G/H with η. This identification of $T_{[g]}(G/H)$ with η is unique up to the adjoint action of H on η.

Proposition 4

If G is a matrix Lie group and H is a connected compact subgroup of G, there exists a volume form on G/H that is invariant under the left action of G. This form is unique up to multiplication by a constant.

Definition

Let T be a fixed maximal torus[a] in G. Let

$$\Phi : T \times (G/T) \to G \tag{27}$$

be defined[b] by

$$\Phi(t, [g]) = gtg^{-1}$$

where $[x]$ denotes the coset gT in G/T.

[a] A subgroup T of G is a torus if T is isomorphic to $(S_1)^k$ for some k. A subgroup T of G is a maximal torus if it is a torus and is not properly contained in any other torus in G.

[b] Let T be a fixed maximal torus in G. Then every $g \in G$ can be written in the form

$$g = hth^{-1}$$

for some $h \in G$ and $t \in T$.

It is simple to see that if $s \in T$, then $(gs)t(gs)^{-1} = gtg^{-1}$.

Compact Group Integration

Proposition 5

Let $(t,[g])$ be a fixed point in $T \times G/T$. If we identify the tangent spaces to $T \times G/T$ and to G with $\mathcal{L}(T) \oplus \mathcal{L}(G/T) \cong \mathcal{L}(G)$ then the differential of Φ at $(t,[g])$ is represented by the following operator

$$\Phi_* = (Ad_g) \begin{pmatrix} I & 0 \\ 0 & Ad_{t^{-1}} - I \end{pmatrix} \tag{28}$$

Compact Group Integration

Proof of Weyl Integral Formula. So, using (18)

$$k \int_G f(g)dg = \int_{T\times G/T} \Phi^*(f(g)dg) \tag{29}$$

$$= \int_{T\times G/T} (f\circ\Phi)\Phi^*(dg)\,, \tag{30}$$

for any smooth function f. Thus, we need to show that
$\Phi^*(dg) = J(t)d[g]\wedge dt$.
Pick orthonormal bases such that

$$\alpha_1(H_1,\ldots,H_r) = \alpha_2(X_1,\ldots,X_N) = \beta(H_1,\ldots,H_r,X_1,\ldots,X_N) = 1 \tag{31}$$

so that

$$(\alpha_1\wedge\alpha_2)(H_1,\ldots,H_r,X_1,\ldots,X_N) \tag{32}$$

Compact Group Integration

Now, with respect to the chosen bases

$$\begin{aligned}
&\Phi^*(\beta)(H_1, \ldots, H_r, X_1, \ldots, X_N) \\
&\qquad = \beta(\Phi_*(H_1), \ldots, \Phi_*(H_r), \Phi_*(X_1), \ldots, \Phi_*(X_N)) \\
&\qquad\quad = det(\Phi_*)(\beta)(H_1, \ldots, H_r, X_1, \ldots, X_N) \\
&\qquad\qquad = J(t)(\alpha_1 \wedge \alpha_2)(H_1, \ldots, H_r, X_1, \ldots, X_N)
\end{aligned}$$

where $J(t) = \det(Ad_{t-1} - I)$ So, up to a constant

$$k \int_G f = C \int_{T \times G/T} J(t) f(gtg^{-1}) d[g] dt \,. \quad \square \tag{33}$$

Compact Group Integration

Corollary 1

If f is a continuous class function on G, then

$$\int_G f = k \int_T J(t) f(t) dt \,. \qquad (34)$$

Proof. If f is a continuous class function, thet $f(gtg^{-1}) = f(t)$ for all $g \in G$ and $t \in T$. It follows that

$$\int_G f(gtg^{-1}) d[g] = f(t) \,. \quad \square \qquad (35)$$

Def.: A class function $f : G \to \mathbb{C}$ is called a *class function* if $f(gtg^{-1}) = f(t)$ for all $g, h \in G$.

SU(2) Example

Suppose G $=$ SU(2) and T is the diagonal subgroup. Then, Corollary 1 take the form

$$\int_{SU(2)} f(g)dg = \frac{1}{2}\int_{-\pi}^{\pi} f(diag(e^{i\theta}, e^{-i\theta}))4\sin^2(\theta)\frac{d\theta}{2\pi}\,, \tag{36}$$

where constant k related to Weyl group is $|W| = 2$ and the normalized volume measure on T is

$$\frac{d\theta}{2\pi}\,.$$

A formula for integration on U(n)

Theorem

If $f : U(n) \to \mathbb{C}$ is a class-function, then

$$\int_{U(n)} f = \frac{1}{n!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(u_1, \ldots, u_n) \prod_{i<j} |u_i - u_j|^2 \frac{d\theta_1}{2\pi} \cdots \frac{d\theta}{2\pi}, \qquad (37)$$

where $u_k = e^{i\theta}$.

In essence, we have to show that

$$J(t) = \prod_{i \neq j} (u_k u_j^{-1} - 1) = \prod_{j<k} |u_j - u_k|^2 . \qquad (38)$$

A formula for integration on U(n)

The other observation is that $J(t) = |\Delta(t)|^2$, where

$$\Delta(t) = \prod_{j<k} (u_j - u_k) \tag{39}$$

is the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_n \\ u_1^2 & u_2^2 & \cdots & u_n^2 \\ \vdots & \vdots & & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} \end{vmatrix} \tag{40}$$

A formula for integration on U(n)

Expanding the determinant

$$\Delta(t) = \sum \pm u_1^{m_1} u_2^{m_2} \cdots u_n^{m_n} \,, \tag{41}$$

where $(m_1, \cdots, m_n)$ runs through the permutations of $(0, 1, \ldots, n-1)$, and integrating each of the $(n!)^2$ terms separately, we find the normalization to be $(2\pi)^n n!$.

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It can be made more explicit the Fourier method to representation theory of Lie groups in order to establish the existence and uniqueness (up to constant multiplication) of Haar invariant measure, however it will be in other chance!. [2, 4]

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THANK YOU!

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