§ Compact Subgroups of Lie Groups (Cap XV Hochschild).

We shall require a **fixed point theorem for compact groups** of linear automorphisms.

Let **V** be a **finite-dimensional** **complex** vector space (what if it is real, like $\mathbb{R}^2$?) and let **F** be a **positive definite hermitian form on V×V** (i.e. $F: V\times V \to \mathbb{C}$ s.t. $v \mapsto F(v,w)$ is $\mathbb{C}$-linear for every $w \in V$, $F(w,v) = \overline{F(v,w)}$ $\forall v,w \in V \Rightarrow w \mapsto H(v,w)$ is $\mathbb{C}$-antilinear $\forall v \in V$, $F(v,v) \in \mathbb{R}$ $\forall v \in T$ and $F(v,v) \geq 0$ $\forall v \in T$, $F(v,v) = 0 \iff v = 0$. If $x \in End_{\mathbb{C}}(V,V)$ and x^* the hermitian conjugate of x, which is determined by $F(x(u),v) = F(u, x^*(v))$ $\forall u,v \in V$.

x is **Hermitian** (self-adjoint) if $x = x^*$. If x is a **Hermitian** $\mathbb{C}$-linear endomorphism of V then x is said to be **positive definite** if $F(x(v),v) > 0$ $\forall v \neq 0 \in V$.

Remark i) $F(u,v) = 0$ $\forall u \in V \Rightarrow v = 0$ since taking $u = v$ then $F(v,v) = 0 \iff v = 0$.

ii) $F(x(u),v) = F(u, x^*(v)) \Rightarrow \overline{F(v, x(u))} = \overline{F(x^*(v),u)} \underset{\text{in } \mathbb{C}}{\Rightarrow} F(v,x(u)) = F(x^*(v),u)$

iii) $F(v, x(u)) = F(x^*(v),u) = F(v, x^{**}(u))$ $\forall u,v \in V \Rightarrow F(v, x(u) - x^{**}(u)) = 0$ $\forall u,v \Rightarrow x = x^{**}$.

iv) Let $u \in V$ s.t. $x^*(u) = 0$ then $\forall w \in V$, $F(x(w),u) = F(w, x^*(u)) = F(w,0)$ in particular $w=0$ implies $F(0,0) = 0$ $\therefore$ $F(x(w),u) = 0$ $\forall w \in V$ but since $x \in Aut_{\mathbb{C}}(V)$ $\therefore F(y,u) = 0$ $\forall y \in V \Rightarrow u = 0$

v) Let $u \in V$ then $\forall v \in V$, $F(u - x^*((x^{-1})^*(u)), v) = F(u,v) - F(x^*(x^{-1})^*(u), v) = F(u,v) - F(u, x^{-1}x(v)) = F(u,v) - F(u,v) = 0 \Rightarrow u - x^*((x^{-1})^*(u)) = 0 \Rightarrow x^*$ is surjective and $(x^*)^{-1} = (x^{-1})^* \Rightarrow x^* \in Aut_{\mathbb{C}}(V)$.

If $x \in End_{\mathbb{C}}(V,V)$ self-adjoint is positive definite $\iff \lambda > 0$ $\forall \lambda$ eigenvalue of x. Let $G = Aut_{\mathbb{C}}(V)$ be the set of $\mathbb{C}$-linear automorphism in V and let $\Sigma = \{x \in Aut_{\mathbb{C}}(V) \mid x = x^* \text{ and positive defined}\}$.

If $x \in G$ and $h \in \Sigma$ define $G \times \Sigma \to \Sigma$ by $(x,h) \longmapsto x \cdot h := x h x^*$.

Remark $Aut_{\mathbb{C}}(V)$ acts on Σ be the right, since $x \cdot h = x h x^*$. i) If $x \in Aut_{\mathbb{C}}(V)$ then $x^* \in Aut_{\mathbb{C}}(V)$, therefore if $h \in \Sigma$ then $h \in Aut_{\mathbb{C}}(V) \Rightarrow x \cdot h = x h x^* \in Aut_{\mathbb{C}}(V)$. ii) $(x h x^*)^* = x^{**} h^* x^* = x h x^*$ since $h \in \Sigma$ then $x h x^*$ is hermitian. iii) $F(x h x^*(v), v) = F(h x^*(v), x^*(v)) \geq 0$ since $x^* \in Aut_{\mathbb{C}}(V)$ and h is positive defined. This implies $x \cdot h \in \Sigma$.

A map $\eta: [0,1] \to \Sigma$ is called **a geodesic arc in Σ** if there is $x \in G$ and $H \in End_{\mathbb{C}}(V)$

hermitian, such that $\eta(r) = x \cdot \mathrm{Exp}(rH)$ for every $r \in [0,1]$

<u>Remark</u> if $H \in \mathrm{End}_{\mathbb{C}}(V)$ hermitian and $r \in [0,1] \subseteq \mathbb{R}$ then $rH \in \mathrm{End}_{\mathbb{C}}(V)$ hermitian since $[\mathrm{Exp}(rH)]^* = \left(\sum_{n=0}^{\infty} \frac{1}{n!}(rH)^n\right)^* = \sum_{n=0}^{\infty} \frac{1}{n!}(rH^n)^* = \sum_{n=0}^{\infty} \frac{r^n}{n!} H^n = \mathrm{Exp}(rH)$. Even more $\mathrm{Exp}(rH)^{-1} = \mathrm{Exp}(-rH)$ then $\mathrm{Exp}(rH) \in \mathrm{Aut}_{\mathbb{C}}(V)$ and also $\mathrm{Exp}(rH)$ is positive since it eigenvalues are given by $\exp(r\lambda) > 0$. with λ eigenvalue of H. [Teoria espectral de operadores para los temas de convergencia].

$\therefore$ $\mathrm{Exp}(rH) \in \Sigma$ and then $x \cdot \mathrm{Exp}(rH)$ is well-defined.

<u>Lemma</u> For every pair $(h_0, h_1) \in \Sigma \times \Sigma$, there is one and only one geodesic arc η in Σ such that $\eta(0) = h_0$, $\eta(1) = h_1$.

<u>Proof</u> $S = \{x \in \mathrm{End}_{\mathbb{C}}(V) \mid x \text{ is self-dual}\}$ is a <u>real</u> vector space. Σ is open in S and $\mathrm{Exp}: S \to \Sigma$ is an analytic map which is invertible with inverse $\mathrm{Log}: \Sigma \to S$ given by $h \mapsto \log(\mathrm{Tr}(h)) I_V - \sum_{n=1}^{\infty} \frac{1}{n}(I_V - \mathrm{Tr}(h)^{-1} h)^n \in S$ (Taylor series around $\mathrm{Tr}(h) I_V$).

For $h \in \Sigma$ (h self-dual) we define $h^{1/2} := \mathrm{Exp}(\frac{1}{2}\mathrm{Log}(h))$, $h^{-1/2} := \mathrm{Exp}(-\frac{1}{2}\mathrm{Log}(h))$, and the geodesic arc

$$\eta(r) := h_0^{1/2} \cdot \mathrm{Exp}(r \,\mathrm{Log}(h_0^{-1/2} h_1 h_0^{-1/2}))$$

which satisfy $\eta(0) = h_0$, $\eta(1) = h_1$. it is the unique geodesic arc s.t. $\eta(0) = h_0$ and $\eta(1) = h_1$.

<u>Definition</u> A subset $E \subseteq \Sigma$ is called **totally geodesic** if, for every $(h_0, h_1) \in E \times E$, the geodesic arc s.t. $\eta(0) = h_0$, $\eta(1) = h_1$ also satisfy $\eta(I) = \mathrm{Im}(\eta) \subseteq E$.

<u>**Theorem**</u> (Totally geodesic Compact fixed point). Let E be a **non-empty closed** totally geodesic subset of Σ, and let K be a **compact** subgroup of $GL(V)$ such that E is (right) K-invariant. Then the action of K in E **has a fixed point in E.**

<u>Idea</u> Define $Q: \Sigma \times \Sigma \to \mathbb{R}$ by $Q(h_1, h_2) = \mathrm{Tr}(h_1 h_2^{-1}) + \mathrm{Tr}(h_2 h_1^{-1})$

<u>Remark</u> By properties of Tr we have:

i) $Q(x \cdot h_1; x \cdot h_2) = \mathrm{Tr}(x h_1 x^* [x h_2 x^*]^{-1}) + \mathrm{Tr}(x h_2 x^* (x h_1 x^*)^{-1})$

$= \mathrm{Tr}(x h_1 x^* x^{*-1} h_2^{-1} x^{-1}) + \mathrm{Tr}(x h_2 x^* x^{*-1} h_1 x^{-1}) = \mathrm{Tr}(x h_1 h_2^{-1} x^{-1}) + \mathrm{Tr}(x h_2 h_1^{-1} x^{-1})$

$= \mathrm{Tr}(h_1 h_2^{-1} x^{-1} x) + \mathrm{Tr}(h_2 h_1^{-1} x^{-1} x) = Q(h_1, h_2)$.

ii) $Q(h_1, h_2) = \mathrm{Tr}(h_1 h_2^{-1}) + \mathrm{Tr}(h_2 h_1^{-1}) = \mathrm{Tr}(h_1 h_2^{-1/2} h_2^{-1/2}) + \mathrm{Tr}(h_2 h_1^{-1/2} h_2^{-1/2})$

$= \mathrm{Tr}(h_2^{-1/2} h_1 h_2^{-1/2}) + \mathrm{Tr}(h_1^{-1/2} h_2 h_1^{-1/2})$ since $h_i^{-1/2} \in \Sigma$ (what happened with $h_i^{-1/2}$?)

$= \mathrm{Tr}(h_2^{-1/2} h_1 (h_2^{-1/2})^*) + \mathrm{Tr}(h_1^{-1/2} h_2 (h_1^{-1/2})^*)$

$= \mathrm{Tr}(h_2^{-1/2} \cdot h_1) + \mathrm{Tr}(h_2^{-1/2} \cdot h_1) \geq 0$ because $h_i^{-1/2} \cdot h_1 \in \Sigma$.

Claim. If η is a geodesic arc in Σ, the function $r \mapsto Q(h, \eta(r))$ with $h \in \Sigma$ is strictly convex if η is not constant. (Prove over $\eta(r) = \mathrm{Exp}(rH)$, $H \in S$.

Lemma Let P be a **compact subset of** Σ, and let $M > 0$. Let also

$$P(M) := \{ h \in \Sigma \mid \inf_{p \in P} \{ Q(p,h) \} \leq M \}$$

then **$P(M)$ is compact.**

Idea. Q is continuous and $P(M) = (\inf_{p \in P} Q(p, \_))^{-1}([0, M])$. Also let $P' = \{ u p u' \mid p \in P$ and $u \in U(V, F) \}$, then $P' \subseteq \Sigma$ is compact, $P \subset P' \Rightarrow P(M) \subseteq P'(M)$ and $P'(M)$ is compact.

↳ unitary automorphisms of V under F.

Now let $h, h_0 \in \Sigma$ and $K \leq G$ compact, define the function $f_{h, h_0} : K \to \mathbb{R}$ by $f_{h,h_0}(k) = Q(h, k \cdot h_0)$, now using the normalized Haar integral on K, $\widetilde{\omega}_K$, define $F_{h_0} : \Sigma \to \mathbb{R}$ by

$$F_{h_0}(h) := \int_K f_{h,h_0}(k) \widetilde{\omega}_K.$$

if $h_0, h \in E$ then $f_{h,h_0}(k) = Q(h, k \cdot h_0)$ ↳ E K-inv.
$= Q(h, h_0) \Rightarrow F_{h_0}(h) = \int_K f_{h,h_0}(k) \widetilde{\omega}_K$
$= Q(h, h_0) \leq M$ $\forall h \in E \Rightarrow \inf_{k \in K} \{ f_{h,h_0}(k) \} = \inf_{k \in K} Q(h, h_0) = Q \leq M$

continuous action.

Remark. Since K is compact, **$K \cdot h_0$ is a compact subset of $\Sigma \Rightarrow (K \cdot h_0)(M)$ is compact.** which means that $\{ h \in \Sigma \mid \inf_{k \in K} \{ f_{h,h_0}(k) \} \leq M \}$ is compact. $(Q(h_1, h_2) = Q(h_2, h_1))$.

Take $h_0 \in E \subseteq \Sigma$, then $\{ h \in E \mid F_{h_0}(h) \leq M \} \subseteq \{ h \in \Sigma \mid \inf_{k \in K} \{ f_{h,h_0}(k) \} \leq M \}$ is a relative closed subset and therefore compact. **Hence there is $h_1 \in E$ for which $F_{h_0}(h_1)$ is minimal**, even more **$x \cdot h_1 = h_1$ $\forall x \in K$.**

§ Application to finite dimensional semisimple Lie algebra over $\mathbb{C}$ with a conjugation σ.

Let L a **finite dimensional semisimple Lie algebra over $\mathbb{C}$ with a conjugation σ.**

↳ It has no non-zero abelian ideals. $\Leftrightarrow \bigoplus V$ simple.

There is another conjugation τ of V such that $\tau \circ \sigma = \sigma \circ \tau$ and L_τ is of **compact type.**

↳ τ-fixed part of L.

Let B denote the trace form of the adjoint representation of V ($B(x,y) = \underset{\text{Killing Form}}{Tr}(ad_x\, ad_y)$) ($ad_x \in End_{\mathbb{C}}(V)$). ($L$ semisimple $\Rightarrow$ B non-degenerated)

and define $F: V\times V \to \mathbb{C}$ by $F(x,y) = -B(x, \tau(y))$

<u>Remark</u> i) $B|_{L_\tau \times L_\tau}$ is negative definite $\Rightarrow$ $F(x,x) > 0$ $\forall\, x \neq 0 \in L_\tau$.

ii) $B(L_\sigma \times L_\sigma) \subseteq \mathbb{R}$, and considering that $L = L_\tau + iL_\tau$ then $B(\tau(x), \tau(y)) = \overline{B(x,y)}$

$\forall x,y \in L$, therefore $F(x,y) = -B(x,\tau(y)) = -B(\tau(y), x) = -\overline{B(y,\tau(x))} = \overline{F(y,x)}$, i.e.

F is Hermitian on $L\times L$.

iii) If $x,y \in L_\tau$, $F(x+iy, x+iy) = -B(x+iy, x-iy) = -B(x,x) - B(y,y) \geq 0$ $\Rightarrow$

F is positive definite on $L\times L$.

Let $A = \{ f \in Aut_{\mathbb{R}\text{-Lie}}(L) \mid f(L_\sigma) \subset L_\sigma \}$ and think L as a real Lie algebra also semi-simple.

<u>Remark</u> $A \subseteq Aut_{\mathbb{R}\text{-Lie}}(L)$ is relative closed $\Rightarrow$ A is a Lie group and $Lie(A) \subseteq Der(L_{\mathbb{R}})$.

<u>Remark</u> Since L is semisimple $\Rightarrow$ $\delta \in Der_{\mathbb{R}}(L)$ is of the form $\delta = D_x$ ($D_x(a) := [x,a]$).

with $x \in L$. If $D_x \in Lie(A)$ $\Rightarrow$ $D_x(L_\sigma) \subset L_\sigma$ $\Leftrightarrow$ $x \in L_\sigma$ $\Rightarrow$ $Lie(A) = \{ \delta \in Der_{\mathbb{R}}(L) \mid \delta = D_x,$

$x \in L_\sigma$ and $x \mapsto D_x$ is an isomorphism of $L_\sigma \xrightarrow{\cong} Lie(A) \}$.

<u>Remark</u> A is $*$-invariant (invariant under adjoint conjugation with respect to F).

Let $U = \{ f \in GL_{\mathbb{C}}(L) \mid f^{-1} = f^* \text{ respect to } F \}$ is a compact subgroup of $GL_{\mathbb{C}}(L)$ (↳ as $\mathbb{C}$-vector space)

<u>Remark</u> $f \in U \Leftrightarrow f\circ\tau = \tau\circ f$. $\Rightarrow$ $A\cap U = \{ f \in A \mid f\circ\tau = \tau\circ f \}$ $\Rightarrow$ $Lie(A\cap U) =$

$\{ D_x \in Der_{\mathbb{R}}(L) \mid x \in L_\sigma$ and $D_x \circ \tau = \tau \circ D_x \}$. $\Rightarrow$ $x = \tau(x)$ $\Rightarrow$ $Lie(A\cap U) = \{ D_x \in Der_{\mathbb{R}}(L) \mid$

$x \in L_\sigma \cap L_\tau \}$.

<u>Def</u> $G := GL_{\mathbb{C}}(L)$ and $\Sigma = \{ g \in GL_{\mathbb{C}}(L) \mid g = g^*$ and g is positive defined (respect to F) $\}$

<u>Remark</u> $\gamma\gamma^* \in \Sigma$ $\forall \gamma \in G$, and if $\delta = Exp(\frac{1}{2} Log(\gamma\gamma^*)) \in \Sigma$ then $(\delta^{-1}\gamma)^*(\delta^{-1}\gamma) =$

$\gamma^* \delta^{-2} \gamma = I_L$ $\Rightarrow$ $\delta^{-1}\gamma \in U$. Also $\gamma = \delta(\delta^{-1}\gamma)$ with $\delta \in \Sigma$ and $\delta^{-1}\gamma \in U$. Hence the

multiplication map $\Sigma \times U \to G$ is surjective and, since $(\Sigma^{-1}\Sigma) \cap U = (1)$ then $\Sigma\times U \to G$ is biyective

and an analytic map. Also, the map $\gamma \mapsto Exp(\frac{1}{2} Log(\gamma\gamma^*))$ is analytic so $\Sigma\times U \to G$ is an

analytic manifold isomorphism. $\Sigma \times U \cong G$

Let $S = \{ g \in GL_{\mathbb{C}}(L) \mid g = g^* \}$ as a $\mathbb{R}$-vector space, then $Exp: S \to \Sigma$ is an analytic manifold isomorphism. Moreover $S \cong \mathrm{Lie}(G)$ (as $\mathbb{R}$-vector spaces) and Exp is the exponential map $\exp: \mathrm{Lie}(G) \to G$. This way we call **$\Sigma$ an exponential manifold factor of G.**

($\Sigma \cong \mathrm{Exp}(S)$ as analytic manifolds).

<u>Remark</u> Σ is $R_{U(1)}^{-1}$-invariant since $g \in \Sigma$ and $f \in U$ then $(fgf^{-1})^* = fgf^*$ (in U, $f^{-1} = f^*$) and is positive defined since $F(fgf^*(x), x) = F(gf^*(x), f^*(x)) \geq 0$ since f is invertible.

§ For the Adjoint Image of a Semisimple Lie Group.

Let **$E := \mathrm{End}_{\mathbb{C}}(L)$ as a $\mathbb{R}$-space.**

<u>Def</u> We say that a subgroup H of $G = GL_{\mathbb{C}}(L)$ is a **real algebraic subgroup of G** if there is a set P of polynomial functions on E such that **H** consists of all those **invertible elements of E which are zeros of P.**

<u>Remark</u> G is a real algebraic subgroup of G and **A is a real algebraic subgroup of G.**

Let T be any subgroup of $GL_{\mathbb{C}}(L)$ that is contained in a real algebraic subgroup H of G such that **H is $*$-invariant** and $T_1 = H_1$ (connected components of I).

Since $T \subseteq G \cong \Sigma \times U$ then given $t \in T$ exist $s \in \Sigma$ and $u \in U$ s.t. $t = su$ then $tt^* = suu^*s^* = s^2 \in \Sigma = \mathrm{Exp}(S)$ then $s^2 = \mathrm{Exp}(x)$, $x \in S$. $s^{2m} \in H$ $\forall m \in \mathbb{Z}$? $tt^* \in H \Rightarrow s^2 \in H \Rightarrow s^{2m} \in H$ $\forall m \in \mathbb{Z}$.

Choose a $\mathbb{C}$-basis for L for which x is diagonal, then if p is a polynomial function on E then $p(\mathrm{Exp}(rx)) = \sum_{i=1}^{q} c_i e^{rk_i}$ $\forall r \in \mathbb{R}$, where $c_i \in \mathbb{R}$ independ of r and k_i's are **distinct real** numbers independent of r. If p vanishes on H then $p(s^{2m}) = 0 \Rightarrow \sum c_i (e^{2k_i})^m = 0$ $\forall m \in \mathbb{Z}$. Letting $d_i = e^{2k_i}$ then $d_1, \dots, d_q$ are q distinct non-zero real numbers so that the determinant with entries d_i^j ($j = 1, \dots, q$ as powers to d_i) is different from 0. $\Rightarrow$ (if two columns in det are equal then $d_i = d_j$ for some $i \neq j$ #)

$c_i = 0$ so that $p(\mathrm{Exp}(rx)) = 0$ $\forall r \in \mathbb{R}$. Hence, since H is an algebraic subgroup of G, we have $\mathrm{Exp}(rx) \in H \cap \Sigma$ $\forall r \in \mathbb{R}$, i.e. **if $\mathrm{Exp}(x) \in H \cap \Sigma$ with $x \in S$ then $\mathrm{Exp}(rx) \in H \cap \Sigma$ $\forall r \in \mathbb{R}$.** This implies that $H \cap \Sigma = (H \cap \Sigma)_1 \subset H_1 \cap \Sigma = T_1 \cap \Sigma \subset T \cap \Sigma \subset H \cap \Sigma$

(if $h \in \Sigma$ then $hh^* = I$ which gives the path from h to I.)

so **$H \cap \Sigma = T \cap \Sigma = (T \cap \Sigma)_1 = \mathrm{Exp}(S_H)$** where S_H is a connected subset of S that is invariant under $\mathbb{R}$-scalar multiplication.

If $x \in S$ s.t. $\mathrm{Exp}(x)=s^2$ then $\frac{1}{2}x \in S_H$, $s=\mathrm{Exp}(\frac{1}{2}x) \in \mathrm{Exp}(S_H) = T \cap \Sigma$ and $t = su$
↳ then $s^{-1} \in T$
where $u = s^{-1}t \in T \cap U \Rightarrow$ $T = (T\cap\Sigma)(T\cap U)$

In particular, when $H = A$ this holds.

Remark. $(A\cap U)_1$ is the analytic subgroup of A whose Lie algebra consist of derivations D_x with $x \in L_\sigma \cap L_\tau$. On the other hand $A \cap \Sigma = \mathrm{Exp}(S_A)$ and since S_A is $\mathbb{R}$-scalar-invariant then $S_A \subset \mathrm{Lie}(A) = D_{L_\sigma}$ thus $S_A = S \cap D_{L_\sigma}$ so that $A\cap\Sigma$ consist of the elements $\mathrm{Exp}(D_x)$ where $x \in L_\sigma$ and $D_x = D_x^* \Rightarrow F(D_x(y),z) = F(y, D_x(z))$ $\forall y,z \in L$. $\Rightarrow$
$B([x,y],\tau(z)) = B(y,\tau([x,z])) = B(y,[\tau(x),\tau(z)]) \Rightarrow [x,y] = -[\tau(x),y]$ $\forall y \in L$
↳ def of B and Jacobi identity
i.e. $x = -\tau(x)$ $\therefore$ $x \in iL_\tau$ thus $S_A = D_{L_\sigma \cap (iL_\tau)}$ and $T\cap\Sigma = A\cap\Sigma = \mathrm{Exp}(D_{L_\sigma\cap iL_\tau})$

This means that the multiplication map [i] $(T\cap\Sigma)\times(T\cap U) \to T$ is an **isomorphism of disjoint unions of analytic manifolds** (the factor $T\cap U$ need not be connected, but since **it is a compact Lie group it has only a finite number of connected components**).

Also $T\cap\Sigma$ is an exponential manifold factor of T_1 and, as a disjoint union of analytic
↳ why not T?
manifolds, T is the cartesian product of $T\cap\Sigma$ and the compact subgroup $T\cap U$.

Furthermore, [ii] $T\cap\Sigma$ is $\mathbb{R}^{-1}L(T\cap U)$-invariant. (remember Σ is $\mathbb{R}^{-1}L(U)$-invariant).

§ A MORE GENERAL CASE (Adjoint representation of X).

Suppose $L_\sigma = \mathrm{Lie}(X)$ of some Lie group X. The adjoint representation of X, $\mathrm{Ad}: X \to \mathrm{Aut}_{\mathbb{C}\text{-Lie}}(L_\sigma)$ $[\mathrm{Ad} = d(R_g^{-1}\circ L_g)_e]$ extends by $\mathbb{C}$-linearity to a representation of X by Lie algebra automorphisms of the complex Lie algebra $L = L_\sigma \otimes \mathbb{C}$. We can see that $\mathrm{Im}(\mathrm{Ad}) = T$ of $G = GL_{\mathbb{C}}(L)$. Remembering that $A = \{f \in \mathrm{Aut}_{\mathbb{C}\text{-Lie}}(L) \mid f(L_\sigma) \subset L_\sigma\}$ then $T \subset A$ and $\mathrm{Ad}[X_1] \subset T$ where X_1 is the analytic subgroup of G that corresponds to the Lie subalgebra $D_{L_\sigma} \subseteq \mathrm{Lie}(G)$. Since $D_{L_\sigma} = \mathrm{Lie}(A)$ then $T_1 = A_1$. Hence the previous analysis on the structure of T holds here to the adjoint image of X.

Finally, by the <u>Theorem</u> we can show that $T\cap U$ of $\mathrm{Ad}[X]$ contains a conjugate of every compact subgroup of T.

<u>Idea</u> i) $E := T \cap \Sigma$ is a totally geodesic subset of Σ. To see this let $h_0, h_1 \in T \cap \Sigma$ then the geodesic arc from h_0 to h_1 is given by $\eta(r) = h_0^{1/2} \circ \mathrm{Exp}(r \, \mathrm{Log}(h_0^{-1/2} h_1 h_0^{-1/2}))$. Notice that $h_i^{1/2}, h_i^{-1/2} \in T \cap \Sigma$ since $h_i^{\pm 1/2} = \mathrm{Exp}(\pm \frac{1}{2} \mathrm{Log}(h_i)) \in \Sigma$ and $h_i^{\pm 1/2} \in T$ (why?) $\Rightarrow h_i^{\pm 1/2} \in T \cap \Sigma$. $\Rightarrow$ $h_0^{-1/2} h_1 h_0^{-1/2} \in T \cap \Sigma$.

(Notice that, since Log and Exp are isomorphisms and one the inverse of the other then if $r \neq 0$ $r\mathrm{Log}$ is also an isomorphism with inverse $\mathrm{Exp}(\frac{1}{r} \_)$

$$\begin{tikzcd} & S \arrow[dr, "\mathrm{Exp}"] & \\ T \cap \Sigma \arrow[ur, "\pm\frac{1}{2}\mathrm{Log}"] \arrow[rr, "\cong"] & & \Sigma \\ \mathrm{Ad}_{x_i} \arrow[rr, mapsto] & & h_i^{\pm 1/2} \end{tikzcd}$$

then $h_i^{\pm 1/2} = \mathrm{Exp}(\pm \frac{1}{2} \mathrm{Log}(\mathrm{Ad}_{x_i}))$ and since $T \cap \Sigma \subseteq \Sigma$ we have that $h_i^{\pm 1/2} \in T$)

then by the same argument $\mathrm{Exp}(r \, \mathrm{Log}(h_0^{-1/2} h_1 h_0^{-1/2})) \in T \cap \Sigma$ $\forall r \in \mathbb{R}$ $\Rightarrow \eta(r) \in T \cap \Sigma$ $\forall r \in \mathbb{R}$ thus $T \cap \Sigma$ is totally geodesic.

Now let **K be any compact subgroup of T**. If $h \in T \cap \Sigma$, $k \in K \Rightarrow k \cdot h = k h k^* \in A \cap \Sigma$. Since $A \cap \Sigma = T \cap \Sigma$ then by the <u>Theorem</u> $T \cap \Sigma$ is $\mathcal{R}^* \mathcal{L}(K)$-invariant and there is some $h \in T \cap \Sigma$ such that $k h k^* = h$ $\forall k \in K$. Now $h^{1/2} \in T \cap \Sigma$ and $\forall k \in K$ $(h^{-1/2} k h^{1/2})^* = h^{1/2} k^* h^{-1/2} = h^{1/2} (h^{-1} k^{-1} h) h^{-1/2} = (h^{-1/2} k h^{1/2})^{-1}$ so $h^{-1/2} k h^{1/2} \in T \cap U$. Thus $h^{-1/2} K h \subset T \cap U$. iii)

<u>Definition</u> Let G be an analytic group and suppose that, as analytic manifold, $G \cong E \times F$ and there are vector subspaces $S_1, \dots, S_k \subseteq \mathrm{Lie}(G)$ such that $S_i \cap S_j = \{0\}$ and the map $\bigoplus_{i=1}^{k} S_k \longrightarrow E$ given by $(x_1, \dots, x_k) \longmapsto \exp_G(x_1) \cdots \exp_G(x_k)$ **is an analytic isomorphism**. Then we say that **E is an exponential manifold factor of G**

<u>Theorem</u> (Maximal Compact Subgroups of Lie Groups)

Let G be a Lie group such that G/G_1 is finite (G_1 is the connected component of the identity) Then **G has a compact subgroup K** and there is an exponential manifold factor $E = E_1 \times \dots \times E_k$ of G_e, where $E_i = \exp_G(S_i)$, such that the following conditions are satisfied)

i) $k E_i k^{-1} = E_i$ $\forall i$ and $\forall k \in K$, i.e. **E_i is $\mathcal{R}^{-1}\mathcal{L}(K)$-invariant.**

ii) The multiplication **$E \times K \to G$** is an isomorphism of disjoint unions of analytic manifolds.

iii) For every compact subgroup L of G, there is an element $x \in G$ such that $xLx^{-1} \subset K$.

Proof For $k=1$ is the previous discution.

Lemma Let A be a compact subgroup of a semidirect product $V \times_f T$, where V is a vector group and T a compact group. Then there is an element $v \in V$ such that $vAv^{-1} \subseteq T$.

Proof Since $A \subseteq V \times_f T$ then an element $a \in A$ could be writing in the form $a = (pr_1(a), pr_2(a))$ with $pr_1(a) \in V$, $pr_2(a) \in T$ (and by the semidirect product if $x, y \in V \times_f T$ then $xy = (x_1, x_2)(y_1, y_2)$ $:= (x_1 + f_{x_2}(y_1), x_2 y_2)$ where $f: T \to Aut(V)$, V is a vector (abelian-group structure).

If we considered the vector-valued linear form $\int_A \widetilde{\omega}_A : C(A, V) \to V$ defined by

$$\int_A \widetilde{\omega}_A (F) := \int_{x \in A} F(x) \widetilde{\omega}_A \in V \text{ (integration of coordinate functions in a given base } \mathcal{B} \text{ of } V).$$

The left-invariant property of the Haar form $\widetilde{\omega}_A$ implies that

$$\int_{x \in L_b(A) = A} \widetilde{\omega}_A F(x) = \int_{x \in A} L_b^*(F(x) \widetilde{\omega}_A) = \int_{x \in A} F(L_b(x)) L_b^* \widetilde{\omega}_A = \int_{x \in A} F(bx) \widetilde{\omega}_A \quad \forall b \in A$$

If we take $F = pr_1 : V \times_f T \to V$ then the previous integral rewrites as $(x_i := pr_i(x))$

$$\int_{x \in A} x_1 \widetilde{\omega}_A = \int_{x \in A} (bx)_1 \widetilde{\omega}_A = \int_{x \in A} (b_1 + f_{b_2}(x_1)) \widetilde{\omega}_A = \int_{x \in A} b_1 \widetilde{\omega}_A + \int_{x \in A} f_{b_2}(x_1) \widetilde{\omega}_A = b_1 \int_{x \in A} \widetilde{\omega}_A + \int_{x \in A} f_{b_2}(x_1) \widetilde{\omega}_A$$

Now considering that $\int_A \widetilde{\omega}_A = 1$ and $f_{b_2} : V \to V$ a linear transformation then $\int_{x \in A} f_{b_2}(x_1) \widetilde{\omega}_A = f_{b_2}\left(\int_{x \in A} x_1 \widetilde{\omega}_A\right)$, then if $v := -\int_{x \in A} x_1 \widetilde{\omega}_A$ then $-v = b_1 - f_{b_2}(v)$.

Notice also that:

i) $(v_1, e)(v_2, e) = (v_1 + \overset{id_V}{f_e}(v_2), ee) = (v_1 + v_2, e)$,

ii) $(a_1, a_2)(b_1, b_2) = (0, e) \iff (a_1 + f_{a_2}(b_1), a_2 b_2) = (0, e) \iff a_2 b_2 = e,\ a_1 + f_{a_2}(b_1) = 0$
$\iff b_2 = a_2^{-1}$ and $b_1 = -f_{a_2}^{-1}(a_1)$, i.e. $(a_1, a_2)^{-1} = (-f_{a_2}^{-1}(a_1), a_2^{-1})$

iii) $(0, b_2)(v, e)(0, b_2)^{-1} = (0, b_2)(v, e)(-f_{b_2}^{-1}(0), b_2^{-1}) = (0, b_2)(v, e)(0, b_2^{-1})$

$$= (0+f_{b_2}(v), b_2 e)(0, b_2^{-1}) = (f_{b_2}(v) + \cancel{f_{b_2}(0)}^{0}, b_2 b_2^{-1}) = (f_{b_2}(v), e)$$

(i.e.) $(v_1, e)(v_2, t) = (v_1 + f_e(v_2), et) = (v_1 + v_2, t)$.

Therefore, since $b_1 = f_{b_2}(v) - v$, then

$$(b_1, b_2) = (-v + f_{b_2}(v), b_2) = (-v, e)(f_{b_2}(v), b_2) = (-v, e)(f_{b_2}(v), e)(0, b_2)$$

$$= (v,e)^{-1}(0,b_2)(v,e)(0,b_2)^{-1}(0,b_2) = (v,e)^{-1}(0,b_2)(v,e)$$

$\Rightarrow T \ni (0,b_2) = (v,e)(b_1,b_2)(v,e)^{-1} \quad \forall b \in A \Rightarrow vAv^{-1} \subseteq T$.

Lemma (Maximal Compact Subgroups of Lie Groups, case G/D).

Let G be a Lie group such that G/G_e is finite. Suppose that **there is a discrete central subgroup D of G such that Theorem holds for G/D**. Then Theorem holds for G.

Idea: Let $q: G \to G/D$ the canonical quotient homomorphism and E, E_i, S_i, K the objects of Thm for G/D, i.e. $E \times K \to G/D$ is an isomorphism of disjoint union of analytic manifolds and $E = E_1 \times \ldots \times E_k$ where $E_i = \exp_{G/D}(S_i)$ and $K \subseteq G/D$ is compact. Define $H := q^{-1}(E)$, then $q|_H : H \to E$ is a covering space. Since E is simply connected (because $\bigoplus_{i=1}^{k} S_i \cong E$ and S_i are vector spaces?) $q|_H$ is therefore a homeomorphism. Hence H could be endow with the structure of analytic manifold such that $q|_H$ is an analytic isomorphism $H \xrightarrow{\cong} E$ and the manifold topology of H coincides with its topology as subset of G. The differential $q_{*e} : \mathrm{Lie}(G) \to \mathrm{Lie}(G/D)$ is an isomorphism (This is true if G and G/D were connected - Smooth Manifold Lee 21.31 (p 557) and Problem 21-22).

$$\begin{tikzcd}
\mathrm{Lie}(G) \arrow[r, "\cong", "q_*"'] \arrow[d, "\exp_G"'] \arrow[dr, phantom, "//"] & \mathrm{Lie}(G/D) \arrow[d, "\exp_{G/D}"] \\
G \arrow[r, "q"] & G/D = E \times K \\
H \arrow[u, hook] \arrow[r, "q|_H", "\cong"'] & E = \exp_{G/D}(S_1) \cdots \exp_{G/D}(S_k) \arrow[u, hook] \\
\mathrm{Lie}(H) \arrow[u, "\exp"] \arrow[r, "\cong"] & \bigoplus_{i=1}^{k} S_i \arrow[u, "\exp^k_{G/D}"', "\cong"]
\end{tikzcd}$$

Then $H = \exp_G(S_1) \cdots \exp_G(S_k)$. Let $\mathrm{Log}_E = (\exp^k_{G/D})^{-1}$ and is therefore analytic, this implies that $\bigoplus_{i=1}^{k} S_i \to H$ is an analytic manifold isomorphism. Now take $M := q^{-1}(K)$, then M is a closed subgroup of G and thus a Lie group. (Closed subgroup Thm).

Since $D \subset M$ and $EK = G/D$ then $HM = G$. Moreover, if $h_1, h_2 \in H$, $m_1, m_2 \in M$ and $h_1 m_1 = h_2 m_2$ then $q(h_1)q(m_1) = q(h_1 m_1) = q(h_2 m_2) = q(h_2)q(m_2) \Rightarrow q(h_1) = q(h_2)$ and $q(m_1) = q(m_2) \overset{q|_H \cong}{\Rightarrow} h_1 = h_2 \Rightarrow m_1 = m_2$.

$\Rightarrow H \times M \to G$ is bijective, and it maps each connected component of $H \times M$ analytically into a connected component of G. Since this map is bijective, there are maps $pr_1^G: G \to H$ and $pr_2^G: G \to M$ such that the inverse of $H \times M \to G$ is given by $g \mapsto (pr_1^G(g), pr_2^G(g))$. Given an element $g \in G$ we can choose connected neighborhoods V of $q(pr_1^G(g))$ in E and W of $q(pr_2^G(g))$ in K such that the neighborhood VW of $q(g)$ in G/D is evenly covered by q.

Let W^* the connected component of $pr_1^G(g)$ in $q^{-1}(W)$ and let $q|_{W^*}^{-1}: W \to W^*$ (such inverse exist since q is an evenly covering map). If $(VW)^*$ denotes the connected component of g in $q^{-1}(VW)$ then $(VW)^*$ is a neighborhood of g in G, in fact $(VW)^* = q|_H^{-1}(V)W^*$, hence if $x \in (VW)^*$ we have $pr_1^G(x) = q|_H^{-1} pr_1^{G/D} q(x)$ and $pr_2^G(x) = q|_{W^*}^{-1} pr_2^{G/D} q(x)$. This way the map $x \mapsto (pr_1^G(x), pr_2^G(x))$ maps each connected component of G analytically into a connected component of $H \times M$ then $H \times M \to G$ is an isomorphism of disjoint union of analytic manifolds. Also each factor $\exp_G(S_i)$ of H is $R^* L(M)$-invariant. ($\exp_{G/D}(S_i)$ is $R^1 L(K)$-invariant).

Notice also that $q_*(\mathrm{Lie}(M)) \cong \mathrm{Lie}(K)$. The $\mathrm{Lie}(K)$ is the direct sum of its center, Z, and the semisimple ideal $([\mathrm{Lie}(K), \mathrm{Lie}(K)])$, $\mathrm{Lie}(K) \cong Z \oplus ([\mathrm{Lie}(K), \mathrm{Lie}(K)])$.

Let P the analytic subgroup of M, whose Lie algebra is Z the P is the connected component of 1 in the center of M, $\Rightarrow$ P is a closed <u>normal</u> subgroup of M. Let Q the maximal compact subgroup of the abelian analytic group P, then Q is connected (actually a toroid and $R^* L(M)$-invariant $\Rightarrow$ $\mathrm{Lie}(Q)$ is $Ad(M)$-invariant and since $M/M_1 \cong G/G_1$ it is finite. Also M_1 lies in the kernel of the adjoint representation of M on $\mathrm{Lie}(P)$ $\Rightarrow$ $\mathrm{Lie}(P)$ is semisimple as an $R^1 L(M)$-submodule so that exist $R^1 L(M)$-submodule S of $\mathrm{Lie}(P)$ such that $\mathrm{Lie}(P) = \mathrm{Lie}(Q) \oplus S$.

$V = \exp_G(S)$.

Let V denote the analytic subgroup of P whose Lie algebra is S. Then $P = VQ$ and $V \cap Q$ is discrete. Since P/Q is a vector group (because Q is the maximum compact subgroup of P) then considering the canonical homomorphism $V \to P/Q$ we see that $V \cap Q = (1)$ $\Rightarrow$ $V \times Q \to P$ is an isomorphism of analytic groups, and V is a closed normal subgroup of M (V is closed in P and P is closed in M). There is an isomorphism

$\mathrm{Lie}(M_1/P) \cong ([\mathrm{Lie}(K), \mathrm{Lie}(K)]) \cong \mathrm{Lie}(K_1/Z(K_1)) \Rightarrow M_1/P$ is compact $\Rightarrow M_1/V/P/V \cong M_1/P$ and P/V is compact therefore M_1/V is compact and since M/M_1 is finite, this implies that M/V is compact. $\Rightarrow M \cong V \times_\tau T$ where $T = M/V$ is a compact group.

Now let $F = HV$ then ii) $F \times T \to G$ is an isomorphism of disjoint union of analytic manifolds and F is an exponential manifold factor of G_1, $F = \exp_G(S_1) \times \dots \times \exp_G(S_k) \times \exp_G(S)$.

Let $t \in T$ then by construction of V we have i) $tVt^{-1} = V$ thus each factor of F is $R^{-1}L(T)$-invariant

Finally let L be any compact subgroup of G, then $q(L)$ is a compact subgroup of G/D so that there is an element $e \in E$ such that $e\,q(L)\,e^{-1} \subset K$. Choose an element $h \in H$ such that $q(h) = e$, then $hLh^{-1} \subset M = V \times_\tau T$ then there is an element $v \in V$ such that $vhLh^{-1}v^{-1} \in T$

now $v \exp_G(S_i) v^{-1} = \exp_G(S_i)$ (by invariance) so then $vhv^{-1} \in H$. Thus $vh = (vhv^{-1})v \in HV$.

and iii) $vhL(vh)^{-1} \subset T$ ∎

This result says how to proced in the case where $\mathrm{Lie}(G)$ is semisimple, taking D to be the kernel of the adjoint representation $\mathrm{Ad}: G \to \mathrm{Lie}(G)$, i.e. the center of G.

Lemma The conclusion of **Lemma** holds also if D is a compact normal subgroup of G.

Lemma The conclusion of **Lemma** holds also if D is a normal closed vector subgroup of G.

Lemma Let G be a Lie group and suppose $\mathrm{Lie}(G)$ is **not semisimple**, then **G has a non-trivial closed normal subgroup that is either a vector group or a toroid.**

Remark The compact group K of Thm (MCSLG) is **necessarily a maximal compact subgroup of G.** Suppose M is a compact subgroup of G containing K, then there is an element $x \in G$ such that $xMx^{-1} \subset K$. Thus we have $xKx^{-1} \subseteq xMx^{-1} \subseteq K \Rightarrow M = K$.

Theorem Let G be a Lie group such that G/G_1 is finite, and suppose that G has a closed connected normal subgroup N such that G/N is compact. Let L be **any maximal compact subgroup of G, then $G = LN$.**

*Summary.

Every G Lie group (with a finite number of connected components) admits a descomposition $G \stackrel{\hat{=}}{} K \times E$ where K is maximal compact subgroup and $\forall L \subset G$ compact $\exists x \in G$ s.t. $xLx^{-1} \subset K$. - Fixed Point Thm

How we can find such a descomposition? There are two important cases i) $Lie(G)$ is semisimple, ii) $Lie(G)$ is not semisimple.

ii) If $Lie(G)$ is not semisimple then by Lemma G has a non-trivial closed subgroup that is a vector group and then G/V will be the correct space to apply the algoritm. (It will have $Lie(G/V)$ semisimple since $V \subset \ker Ad$.

i) If $Lie(G)$ is semisimple then taken $D = \ker Ad$ then $Ad(G/D) = Lie(G)$ and proving the result over G/D it will be valid for G with a descomposition $F \times T \to G$ where $F = q^{-1}(K)/V$, K the compacts subgroup that satisfies $K \times E \to G/D$ and $V = \exp_G(S)$ where S satisfies $Lie(K) = Z \oplus ([Lie(K), Lie(K)])$, $P \subseteq q^{-1}(K)$, analytic s.t. $Lie(P) = Z$ and since P is closed and normal then there is Q a maximal compact subgroup of P $\Rightarrow Lie(P) = Lie(Q) \oplus S$

Also $T = q^{-1}(K)/V$ is compact then $N \equiv V \rtimes T$.

i') To solve the problem in the final case $Lie(X)$ semisimple and s.t. $Lie(X) = L_\sigma$ (real form) of some lie algebra. Extend $\mathbb{C}$-linearly $Ad: X \to \underset{\mathbb{R}\text{-Lie}}{Aut}(L_\sigma)$ to $Ad: X \to L_\sigma \otimes \mathbb{C} \cong GL_{\mathbb{C}}(L_\sigma \otimes \mathbb{C})$ and take $G = Ad[X]$. Then take the hermitian bilienear for $F_{\tau}(x,y) = -Tr(Ad_x, Ad_{\tau(y)})$ (τ the only conjugation s.t. L_τ is compact); Σ_F the self-dual positive elements of G an U_F the unitary elements of F then $G \cap \Sigma_F \times G \cap U_F \to G$ where $G \cap U_F$ is the desired compact subgroup of G.

Finally, if were are able to find a compact group K' of G such that for any other M compact subgroup of G there is an element $x \in G$ s.t. $xMx^{-1} \subset K'$ then $K' \underset{R'L}{\simeq} K$ of $G \leftarrow K \times E$ (that could be verified via the Thm to find a fixed point of the K' action on some E closed totally geodesic space).

Facts. A Lie algebra is semisimple if it has no non-zero abelian ideals.

- Given L a Lie algebra, it is called solvable if for the sequence $L_0 = L$, $L_{i+1} = ([L_i, L_i])$ then $L_k = \{0\}$ for some k.
- Given L a Lie algebra, there is a unique maximal solvable ideal R of L called its radical, and L/R is semisimple.
- If a Lie algebra L is semisimple then $L = ([L, L])$

$\Rightarrow$ A Lie algebra $L \cong R + S$ with R its radical ideal and $S = \mathrm{Im}(\underset{¿\text{unique}?}{L/R \to L})$.

and $([L,L]) = S + ([L,R])$ since $([S,S]) = S$.

8. Maximal Compact Subgroups

Theorem In a Lie group G with a finite number of connected components there **always exist maximal compact subgroups.** If K is one of them, then any compact subgroup of G is **conjugate** to a subgroup of K, and in particular any two maximal compact subgroups are conjugate. Furthermore **G is homeomorphic to $K \times \mathbb{R}^m$** for some m.

Examples

1) $E_2 := \{ f : \mathbb{R}^2 \to \mathbb{R}^2 \mid \|x-y\|_2 = \|f(x)-f(y)\|_2 \} = \{ \tau \circ A \mid A \in O(2),\ \tau = t_p \text{ for some } p \in \mathbb{R}^2 \}$

Let $f(0)=0$ then $\|x-y\|_2 = \|f(x)-f(y)\|_2$ implies $\|x\|_2 = \|f(x)\|_2\ \forall x. \Leftrightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle$

Let $a_i := f(e_i)$ is a basis (f preserve metric $\Rightarrow$ f preserve inner product $\Rightarrow$ a_i are orthogonal $\Rightarrow$ a_i is a basis of $\mathbb{R}^n$) $\Rightarrow f(x) = \sum \alpha_i(x) a_i \Rightarrow \alpha_j(x) = \langle f(x), a_j\rangle = \langle f(x), f(e_j)\rangle = \langle x, e_j\rangle = x_j \therefore f(x) = \sum x_i a_i = Ax.$

If $f(0) \neq 0 = a$ then $t_{-a} f(0) = t_{-a}(a) = a - a = 0 \therefore t_{-a} f$ preserves metric $\Rightarrow$ $t_{-a} f(x) = Ax \Rightarrow f(x) = Ax + a.$ (Munkres, Analysis on Manifolds 4.20 (20.5) p174).

E_2 acts on $\mathbb{R}^2$, and let H a compact subgroup of E_2, e.g. $H = O(2) \cong S^1$ (any reflection is a rotation in $\mathbb{R}^2$). Now given $x_0 \in \mathbb{R}^2$ consider $R_{x_0} : H \to \mathbb{R}^2$ given by $R_{x_0}(h) := h \cdot x_0 = h(x_0)$

Remark i) E_2 act continuously on $\mathbb{R}^2$ since every coordinate function is polynomial on x (even more smoothly)

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, shift left, "pr_1"] \arrow[d, shift right, "pr_2"'] \\ E_2 \times \mathbb{R}^2 \arrow[ur, "\mu"] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \mathbb{R} \end{tikzcd}$$

$\mu[T, x] = T \cdot x = T(x)$ where $T = t_p \circ A$

then $T(x) = t_p A(x) = (\underbrace{a_{11}x_1 + a_{21}x_2 + p_1}_{f_1(x)}, \underbrace{a_{12}x_1 + a_{22}x_2 + p_2}_{f_2(x)})$

now $E_2 \times \mathbb{R}^2 \xrightarrow{f_1, f_2} \mathbb{R}$ are continuous (even more smooth since it is polynomial).

ii) R_{x_0} is continuous since taking $U \in \tau_{\mathbb{R}^2}$ then $R_{x_0}^{-1}(\mathbb{R}^2 \setminus U) = \{ h \in H \mid R_{x_0}(h) \in \mathbb{R}^2 \setminus U \}$
$= \{ h \in H \mid \mu(x_0, h) \in \mathbb{R}^2 \setminus U \} = \mu^{-1}(\mathbb{R}^2 \setminus U)$
$\cap \{ \{x_0\} \times E_2 \}$ is closed in $E_2 \times \mathbb{R}^2$, or

$$\begin{tikzcd} E_2 \arrow[d, "(id_{E_2}, ct_{x_0})"'] \arrow[dr, bend left, "R_{x_0}"] & \\ E_2 \times \mathbb{R}^2 \arrow[r, "\mu"'] & E_2 \end{tikzcd}$$

L_{x_0}

Therefore $\int_H : C(G,\mathbb{R}^2) \to \mathbb{R}^2$ could be evaluated on R_{x_0} (with the unique normalized Haar volume form $\widetilde{\omega}_H := (\int_H \omega_H)^{-1}\omega_H$ with $\omega_H = \varepsilon^1\wedge\dots\wedge\varepsilon^m$, $\{\varepsilon^i\}$ dual (positively oriented) basis in $\mathfrak{h}^*$ (See Introduction to Smooth Manifolds, Lee 16.10, p 410). This way $\int_H R_{x_0}(h)\widetilde{\omega}_H \in \mathbb{R}^2$, i.e.

$$u = \int_H R_{x_0}(h)\widetilde{\omega}_H \in \mathbb{R}^2$$

Remark since $\int_H R_{x_0}(h)\widetilde{\omega}_H \in \mathbb{R}^2$ then there is $u_1, u_2 : \mathbb{R}^2 \Rightarrow \mathbb{R}$ coordinate functions such that $\int_H R_{x_0}(h)\widetilde{\omega}_H = u_1(x_0)e_1 + u_2(x_0)e_2$ where $u_i(x_0) = \langle \int_H R_{x_0}(h)\widetilde{\omega}_H, e_i \rangle$ also $R_{x_0}(h) = h(x_0) = v_1(h,x_0)e_1 + v_2(h,x_0)e_2$ then $v_i(h,x_0) = \langle R_{x_0}(h), e_i \rangle$
then $u_i(x_0) = \langle \int_H R_{x_0}(h)\widetilde{\omega}_H, e_i \rangle = \langle \int_H (v_1(h,x_0)e_1 + v_2(h,x_0)e_2)\widetilde{\omega}_H, e_i \rangle$

$$= \langle \int_H v_1(h,x_0)\widetilde{\omega}_H\, e_1, e_i \rangle + \langle \int_H v_2(h,x_0)\widetilde{\omega}_H\, e_2, e_i \rangle$$
$$= \int_H v_i(h,x_0)\widetilde{\omega}_H .$$

therefore $u = \int_H (R_{x_0}(h))_1\widetilde{\omega}_H\, e_1 + \int_H (R_{x_0}(h))_2\widetilde{\omega}_H\, e_2$ where $\int_H (R_{x_0}(h))_i\widetilde{\omega}_H \in \mathbb{R}$.

Remark ω_H is left-invariant $\therefore$ $\widetilde{\omega}_H$ is also left-inv. $\Rightarrow \int_H R_{x_0}(kh)\widetilde{\omega}_H = \int_H R_{x_0}(kh) L_k^*\widetilde{\omega}_H$
$= \int_H L_k^*(R_{x_0}(h)\widetilde{\omega}_H) = \int_{L_k(H)} L_k^*(R_{x_0}(h)\widetilde{\omega}_H) \underset{\text{change of variable Thm.}}{=} \int_H R_{x_0}(h)\widetilde{\omega}_H$ since L_k is a diffeomorphism $\forall k \in H$.

Claim The assignment $x \overset{T}{\mapsto} \int_H R_x(h)\widetilde{\omega}_H$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a contraction

Proof $\|\int_H R_x(h)\widetilde{\omega}_H - \int_H R_y(h)\widetilde{\omega}_H\|^2 = \|\int_H (R_x(h) - R_y(h))\widetilde{\omega}_H\|^2 = \|\int_H (h(x) - h(y))\widetilde{\omega}_H\|^2$
$= \|\int_H (A(x)+p - A(y) - p)\widetilde{\omega}_H\|^2 = \|\int_H A(x-y)\widetilde{\omega}_H\|^2 = \|\int_H R_{x-y}(A)\widetilde{\omega}_H\|^2 = |\int_H R_{x-y}(A)_1\widetilde{\omega}_H|^2 +$
$|\int_H R_{x-y}(A)_2\widetilde{\omega}_H|^2 \le 2(\int_H \|R_{x-y}(A)\|\,|\widetilde{\omega}_H|)^2 \le 2M\|x-y\| \cancelto{1}{\int_H \widetilde{\omega}_H} = \alpha\|x-y\|$.

This means via fixed point theorem that $x \mapsto \int_H R_x(h)\widetilde{\omega}_H$ has a fixed point $x_0 \in \mathbb{R}^2$, i.e.
$\int_H R_{x_0}(h)\widetilde{\omega}_H = x_0$.

Claim $\int_H R_{x_0}(h)\widetilde{\omega}_H = x_0 \Rightarrow R_{x_0}(h) = x_0\ \forall h \in H$.

Proof Remember that μ is smooth $\Rightarrow R_{x_0}$ is smooth $\Rightarrow R_{x_0}$ is continuous.

Also $\int_H R_{x_0}(h)_i\widetilde{\omega}_H = x_0^i$

$$\int_H R_{x_0}(h)\widetilde{\omega}_H = \int_H a_h(x_0)\widetilde{\omega}_H = \int_H a_h(x_0) L_h^*\widetilde{\omega}_H = \int_H x_0\widetilde{\omega}_H = x_0$$

↳ L_h diffeo

$F_{h_0}(h) := \int_K \widetilde{\omega}_K f_{h,h_0}(k) = \int_K \widetilde{\omega}_K\, Q(h, x\cdot h_0)$ $F_{h_0}: \Sigma \to \mathbb{R}$ (even more $[0,\infty)$).

$= \int_K \widetilde{\omega}_K \{ \mathrm{Tr}(h(x\cdot h_0)^{-1}) + \mathrm{Tr}(x\cdot h_0 h^{-1})\}$ has a fixed point $h_1 \in H$, i.e.

$F_{h_0}(h_1) = h_1 = \int_K \widetilde{\omega}_K f_{h_1,h_0}(k)$ $h_0, h_1, h \in \Sigma$ and $x \in K \overset{\text{compact}}{\subseteq} GL(V)$

↳ self-adjoint positive defined

if $h_0 \in E$ nonempty totally geodesic subset of Σ $\underline{\underline{\text{closed}}}$

$h_1 := \min_{h \in E} \{F_{h_0}(h)\}$

$E_2 \subseteq GL_{\mathbb{R}}(\mathbb{R}^3) \subseteq GL_{\mathbb{C}}(\mathbb{R}^3)$ via $\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in GL_{\mathbb{R}}(\mathbb{R}^3)$ s.t. $A \in O(2)$ and $v \in \mathbb{R}^2$.

$E = \exp_{E_2}(\mathbb{R}^2)$ is the totally geodesic closed subset of Σ_g then any H compact subgroup acting on E such that E is $R^{-1}L(H)$-invariant then it will have a fixed point in E.

this means that the function $F_{h_0}(h) := \int_H \widetilde{\omega}_H \{ \mathrm{Tr}(h(x\cdot h_0)^{-1}) + \mathrm{Tr}(x\cdot h_0 h^{-1})\}$ has a minimum $h_1 \in E \Rightarrow H \subseteq {E_2}_{h_1}$ (isotropy group of h_1) $\cong t_{h_1} O(2) t_{-h_1}$

Also $\mathrm{Lie}(E_2) \cong T_{I\times 0} E_2 = T_{I\times 0} O(2) \times \mathbb{R}^2 \cong \mathfrak{o}(2) \times \mathbb{R}^2 \cong \mathfrak{so}(2) \times \mathbb{R}^2$ not semisimple