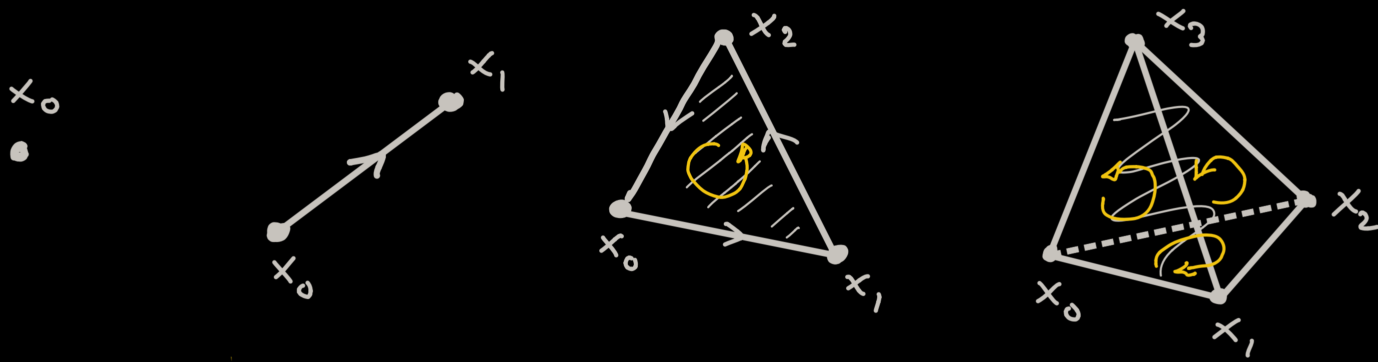
Cap. 7. Grupos de Homología Simplicial.

7.1 Definición de los Gpos. de Homología.

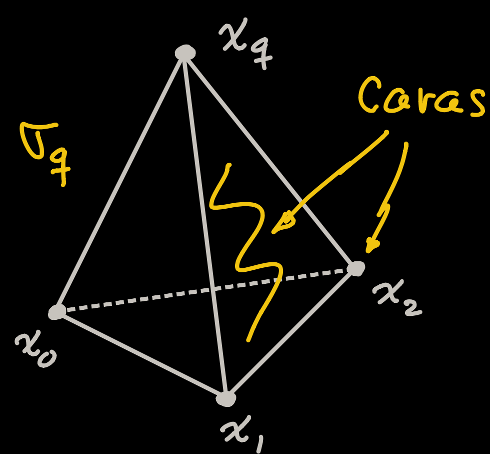
simplejos en $\mathbb{R}^n$:



simplejo de dim q:

$\sigma_q \subseteq \mathbb{R}^n$

= envoltura convexa de $(q+1)$ puntos $x_0, \dots, x_q$ geométricamente indep.



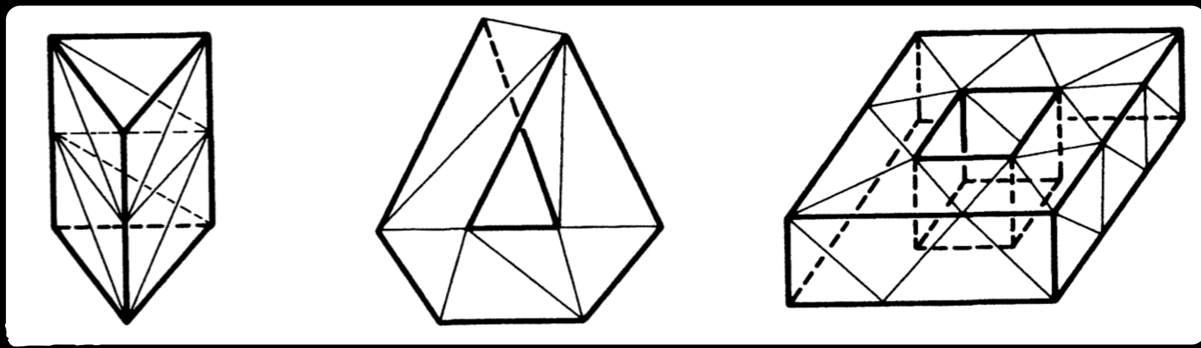
$\iff (x_1 - x_0), \dots, (x_q - x_0)$ l.i.

Def: Un complejo simplicial K es un conjunto finito de simplejos en $\mathbb{R}^n$ tal que:

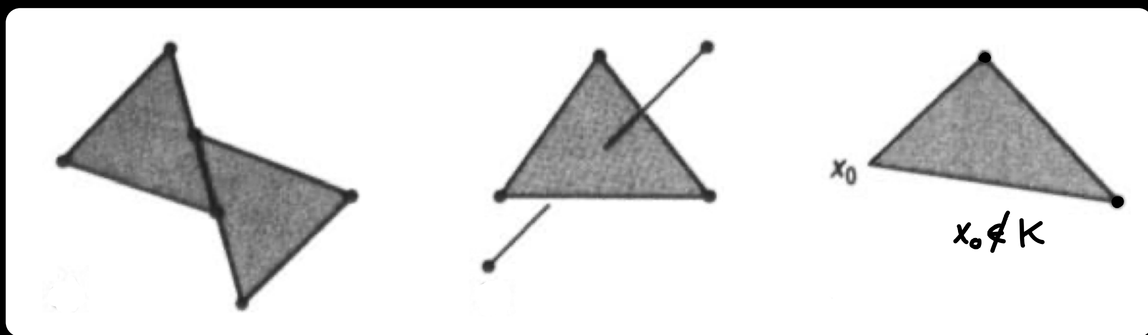
1. $\sigma \in K$ y $\tau < \sigma \Rightarrow \tau \in K$.

2. $\sigma, \tau \in K \Rightarrow \sigma \cap \tau \in K$

Ejems:

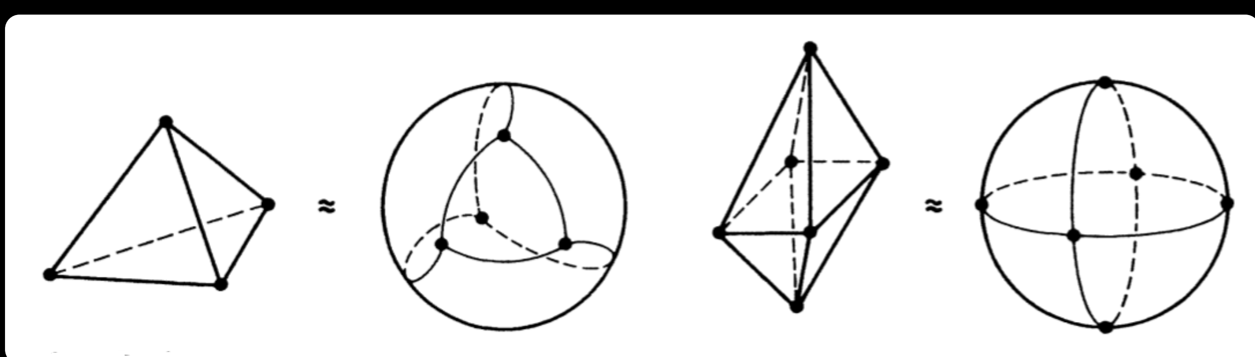


Ejems: No son complejos simpliciales

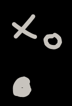


Espacio subyacente: $|K| = \bigcup_{\sigma \in K} \sigma$

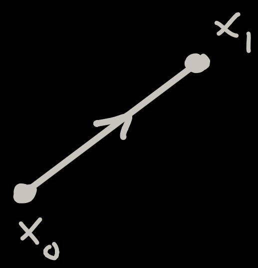
$\left(\begin{array}{l}\text{espacio triangolable} \\ \text{ó poliedro}\end{array}\right)$



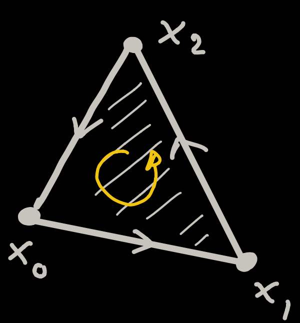
Simplejos orientados:



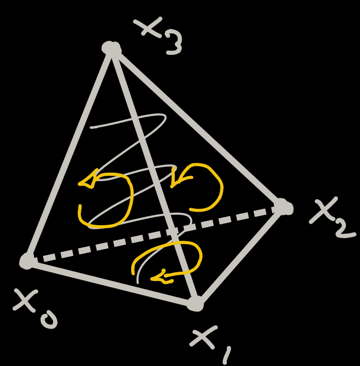
$\langle x_0 \rangle$



$\langle x_0, x_1 \rangle = -\langle x_1, x_0 \rangle$



$$\begin{aligned}\langle x_0, x_1, x_2 \rangle &= \langle x_1, x_2, x_0 \rangle \\ &= \langle x_2, x_0, x_1 \rangle \\ &= -\langle x_1, x_0, x_2 \rangle \\ &= -\langle x_0, x_2, x_1 \rangle \\ &= -\langle x_2, x_1, x_0 \rangle\end{aligned}$$



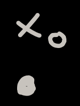
$\langle x_0, x_1, x_2, x_3 \rangle$

Permutaciones pares	Permutaciones impares
= orientación positiva	= orientación negativa

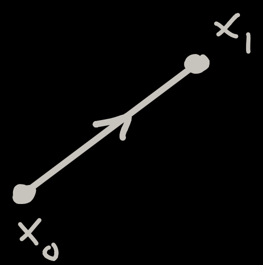
Convención: $\langle x_0, \dots, x_q \rangle = (-1)^{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(q)} \rangle$

$\forall \sigma$ permutación de $\{0, 1, \dots, q\}$

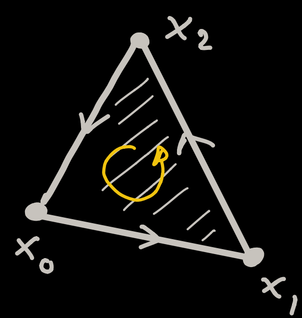
Frontera de un simplejo (en el sentido algebraico)



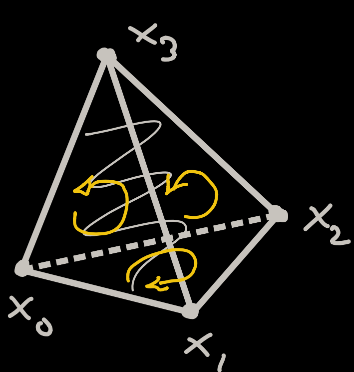
$\partial\langle x_0\rangle = 0$.



$\partial\langle x_0, x_1\rangle = \langle x_1\rangle - \langle x_0\rangle$.



$$\partial\langle x_0, x_1, x_2\rangle = \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_0\rangle$$
$$= \langle x_1, x_2\rangle - \langle x_0, x_2\rangle + \langle x_0, x_1\rangle$$



$$\partial\langle x_0, x_1, x_2, x_3\rangle = \langle x_1, x_2, x_3\rangle + \langle x_0, x_3, x_2\rangle + \langle x_0, x_1, x_3\rangle + \langle x_0, x_2, x_1\rangle$$
$$= \langle x_1, x_2, x_3\rangle - \langle x_0, x_2, x_3\rangle + \langle x_0, x_1, x_3\rangle - \langle x_0, x_1, x_2\rangle$$

En general:

$$\partial\langle x_0, \dots, x_q\rangle = \sum_{i=0}^{q} (-1)^i \langle x_0, \dots, \hat{x}_i, \dots, x_q\rangle$$

se omite x_i

$(-1)^i\langle x_0, \dots, \hat{x}_i, \dots, x_q\rangle$
cara opuesta a x_i con la orientación inducida

Def: K = complejo simplicial. Para $q = 0, 1, 2, \ldots$ definimos las q-cadenas de K:

$$C_q(K) = \text{gpo. abeliano libre generado por los } q\text{-simplejos (orientados) de } K.$$

- Si α_q = núm. de q-simplejos de K, entonces: $C_q(K) \cong \mathbb{Z}^{\alpha_q}$

- Eligiendo una orientación fija para los q-simplejos de K, $\sigma_q^1, \ldots, \sigma_q^{\alpha_q}$; todo elemento de $C_q(K)$ es de la forma:

$$c = n_1 \sigma_q^1 + \ldots + n_{\alpha_q} \sigma_q^{\alpha_q}$$

Def: Si $\sigma_q = \langle x_0, \ldots, x_q \rangle$ q-simplejo orientado de K definimos $\partial \sigma_q$ como la $(q-1)$-cadena:

$$\partial \sigma_q = \partial \langle x_0, \ldots, x_q \rangle = \sum_{i=0}^{q} (-1)^i \langle x_0, \ldots, \hat{x}_i, \ldots, x_q \rangle$$

Para $q=0$: $\partial \langle x_0 \rangle = 0$ Aquí: $C_{-1}(K) = 0$.

Esta fórmula define un homomorfismo de gpos. abelianos:

$$\begin{aligned} \partial_q : C_q(K) &\longrightarrow C_{q-1}(K) \\ \sigma_q &\longmapsto \partial \sigma_q \end{aligned}$$

Sucesión de gpos. abelianos y homomorfismos:

$$\cdots \xrightarrow{\partial_{q+1}} C_q(K) \xrightarrow{\partial_q} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0 = 0} 0$$

Def: $\forall q \geq 0$ ponemos:

$Z_q(K) = \ker \partial_q$ $\quad$ q-ciclos de K.

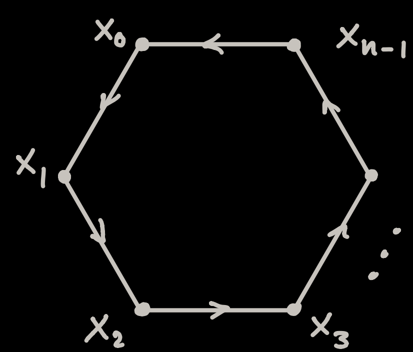
$B_q(K) = \operatorname{im} \partial_{q+1}$ $\quad$ q-fronteras de K.

ambos subgrupos de $C_q(K)$.

Ejems:

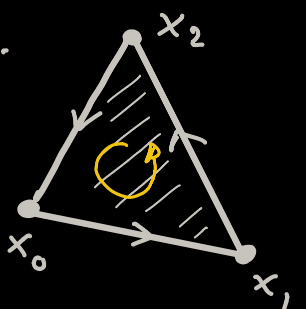
a). $Z_0(K) = \ker \partial_0 = C_0(K)$

b). $c = \langle x_0, x_1 \rangle + \langle x_1, x_2 \rangle + \ldots + \langle x_{n-1}, x_0 \rangle \in C_1(K)$



$\partial c = 0 \quad \Rightarrow \quad c \in Z_1(K)$

c).

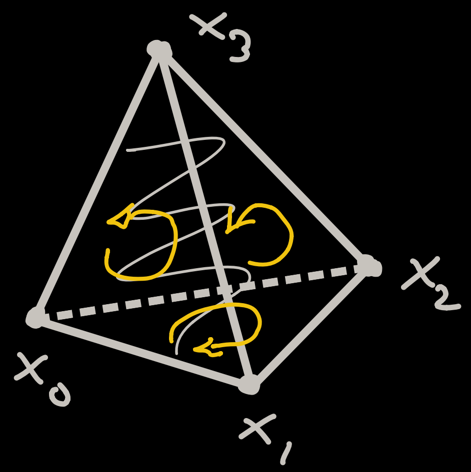


$\partial \langle x_0, x_1, x_2 \rangle = \langle x_0, x_1 \rangle + \langle x_1, x_2 \rangle + \langle x_2, x_0 \rangle$

$\partial \partial \langle x_0, x_1, x_2 \rangle = 0$

$\therefore \quad \partial \langle x_0, x_1, x_2 \rangle \in Z_1(K)$

d). $K =$ superficie de un tetraedro



Sea: $c = \langle x_0, x_1, x_3\rangle + \langle x_1, x_2, x_3\rangle$

$$\partial c = \langle x_0, x_1\rangle + \langle x_1, x_3\rangle + \langle x_3, x_0\rangle + \langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle$$

$$= \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_0\rangle$$

y $\partial(\partial c) = 0$.

Lema: $\partial_q \circ \partial_{q+1} = 0$ $\qquad C_{q+1}(K) \xrightarrow{\partial_{q+1}} C_q(K) \xrightarrow{\partial_q} C_{q-1}(K)$

Dem: $\partial_{q+1}\langle x_0, \dots, x_{q+1}\rangle = \sum_{i=0}^{q+1} (-1)^i \langle x_0, \dots, \hat{x}_i, \dots, x_{q+1}\rangle$

$$\partial_q \partial_{q+1}\langle x_0, \dots, x_{q+1}\rangle = \sum_{i=0}^{q+1} (-1)^i \partial_q \langle x_0, \dots, \hat{x}_i, \dots, x_{q+1}\rangle$$

$$= \sum_{i=0}^{q+1} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j \langle x_0, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_{q+1}\rangle + \sum_{j=i+1}^{q+1} (-1)^{j-1} \langle x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}\rangle \right]$$

$$= \sum_{j<i} (-1)^{i+j} \langle x_0, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_{q+1}\rangle + \sum_{j>i} (-1)^{i+j-1} \langle x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}\rangle = 0.$$

Obs:
$$C_{q+1}(K) \xrightarrow{\partial_{q+1}} C_q(K) \xrightarrow{\partial_q} C_{q-1}(K)$$

$$\partial_q \circ \partial_{q+1} = 0 \quad \Rightarrow \quad \operatorname{im} \partial_{q+1} \leq \ker \partial_q$$

i.e. $B_q(K) \leq Z_q(K)$.

$\therefore$ Toda frontera es un ciclo, pero no necesariamente todo ciclo es una frontera.

Def: K = complejo simplicial, definimos el q-ésimo gpo. de homología de K

$$H_q(K) = Z_q(K) / B_q(K)$$

(ciclos que no son fronteras)

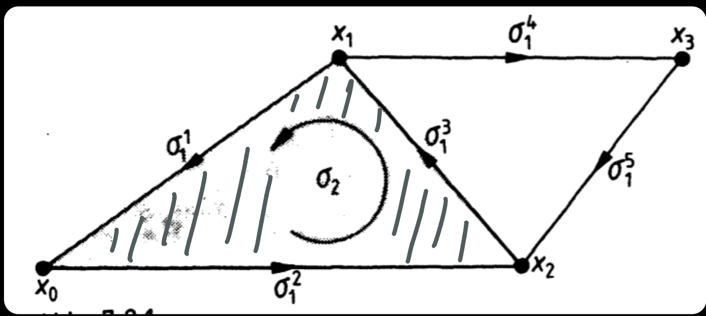
- $H_q(K)$ = gpo. abeliano finitamente generado

$$\cong \underbrace{\mathbb{Z}^{p_q}}_{\text{parte libre}} \oplus \underbrace{\mathbb{Z}/t_q^1 \oplus \cdots \oplus \mathbb{Z}/t_q^{r_q}}_{\text{parte de torsión}}$$

- Rango de $H_q(K) = p_q$ q-ésimo núm. de Betti de K.

7.2. Ejemplos de Gpos. de Homología

Ejem:



0-cadenas:

$$c_0 = a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$$

1-cadenas:

$$c_1 = b_1 \sigma_1^1 + \dots + b_5 \sigma_1^5$$

2-cadenas: $m \cdot \sigma_2$

donde $a_i, b_j, m \in \mathbb{Z}$.

$$0 \xrightarrow{\partial_3 = 0} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0 = 0} 0$$

- Notemos que: $Z_2(K) = \ker \partial_2 = 0$
  $B_2(K) = \operatorname{im} \partial_3 = 0$ $\quad \therefore \quad H_2(K) = 0$

- $\partial c_1 = b_1(x_0 - x_1) + b_2(x_2 - x_0) + b_3(x_1 - x_2) + b_4(x_3 - x_1) + b_5(x_2 - x_3)$

$$= (b_1 - b_2)x_0 + (-b_1 + b_3 - b_4)x_1 + (b_2 - b_3 + b_5)x_2 + (b_4 - b_5)x_3 .$$

$$\partial c_1 = 0 \iff \begin{cases} b_1 = b_2 \\ b_4 = b_5 \\ b_1 = b_3 - b_4 \end{cases}$$

$$Z_1(K): \quad b_1\sigma_1^1 + b_1\sigma_1^2 + (b_1+b_4)\sigma_1^3 + b_4\sigma_1^4 + b_4\sigma_1^5$$

$$= b_1(\underline{\sigma_1^1+\sigma_1^2+\sigma_1^3}) + b_4(\underline{\sigma_1^3+\sigma_1^4+\sigma_1^5})$$

Base para $Z_1(K)$

$z_1 = \sigma_1^1 + \sigma_1^2 + \sigma_1^3$

$z_1' = \sigma_1^3 + \sigma_1^4 + \sigma_1^5$

$$B_1(K) = \{ m(\sigma_1^1+\sigma_1^2+\sigma_1^3) \mid m \in \mathbb{Z} \}$$

$$\therefore \quad H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}$$ generado por $\{z_1'\}$

Un cálculo similar muestra que: $H_0(K) \cong \mathbb{Z}$. (Ejercicio).

$$\therefore \quad H_q(K) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} & q=1 \\ 0 & q\geq 2 \end{cases}$$

Obs: K = complejo simplicial de $\dim(K)=n$.

- Si $q>n$, $H_q(K)=0$.

- $C_{n+1}(K)=0 \Rightarrow B_n(K)=0$, $\therefore H_n(K)=Z_n(K)$.

(generado por los q-ciclos lin. independientes)

• Si $K_1, \dots, K_r$ son las componentes de K, entonces:

$$C_q(K) \cong C_q(K_1) \oplus \dots \oplus C_q(K_r)$$

$$H_q(K) \cong H_q(K_1) \oplus \dots \oplus H_q(K_r).$$

Tma: Para todo complejo simplicial K

$H_0(K) \cong$ gpo. abeliano libre de rango igual al núm. de componentes conexas de K.

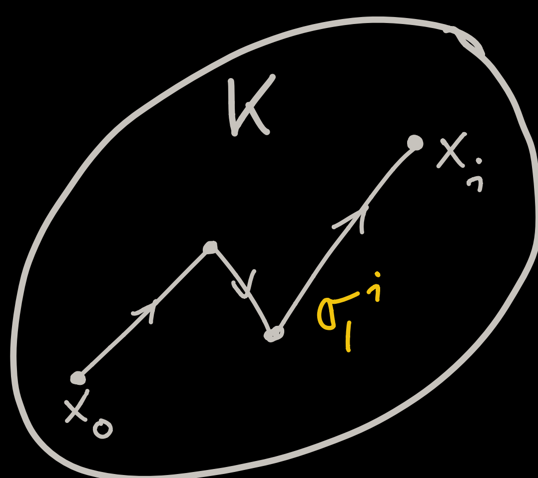
Dem: Basta calcular $H_0(K)$ para K conexo.

$$C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

$$H_0(K) = \frac{Z_0(K)}{B_0(K)} = \frac{C_0(K)}{B_0(K)}$$

En el cociente: $\{c\} = \{c'\} \iff c - c'$ es una frontera

Sean $x_0, \dots, x_n$ los vértices de K



Para $i = 1, \dots, n$ sea σ_1^i una poligonal de x_0 a x_i

Entonces: $\partial \sigma_1^i = x_i - x_0$

$\therefore \quad \{x_i\} = \{x_0\}$ en $H_0(K)$

$$\{m_0 x_0 + \dots + m_n x_n\} = m_0\{x_0\} + \dots + m_n\{x_n\}$$
$$= (m_0 + \dots + m_n)\{x_0\}$$

$$\therefore \quad H_0(K) = \{m\{x_0\} \mid m \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Equivalentemente:

Sea $\varepsilon : C_0(K) \longrightarrow \mathbb{Z}$ dada por

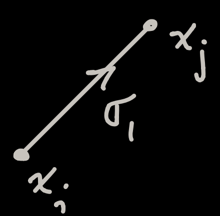
$$\varepsilon(m_0 x_0 + \dots + m_n x_n) = m_0 + \dots + m_n.$$

- ε es un epimorfismo de gpos.

$$\therefore \quad C_0(K)/\ker\varepsilon \cong \mathbb{Z}.$$

Afirmamos que $\ker\varepsilon = B_0(K)$ fronteras en dim. 0

- $B_0(K) \subseteq \ker\varepsilon$



$$\partial\sigma_1 = x_j - x_i$$
$$\Rightarrow \varepsilon(\partial\sigma_1) = 1 - 1 = 0.$$

- $\ker\varepsilon \subseteq B_0(K)$

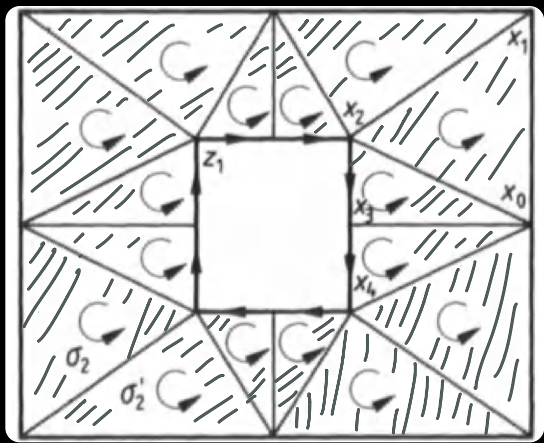
Sea $c = m_0 x_0 + \dots + m_n x_n$ tal que $\varepsilon(c) = 0$.

$$\therefore \quad c = -(m_1 + \dots + m_n)x_0 + m_1 x_1 + \dots + m_n x_n$$
$$= m_1(x_1 - x_0) + \dots + m_n(x_n - x_0) \in B_0(K).$$

∎

Ejem: El cilindro $S^1 \times I$ es homeomorfo a un anillo.

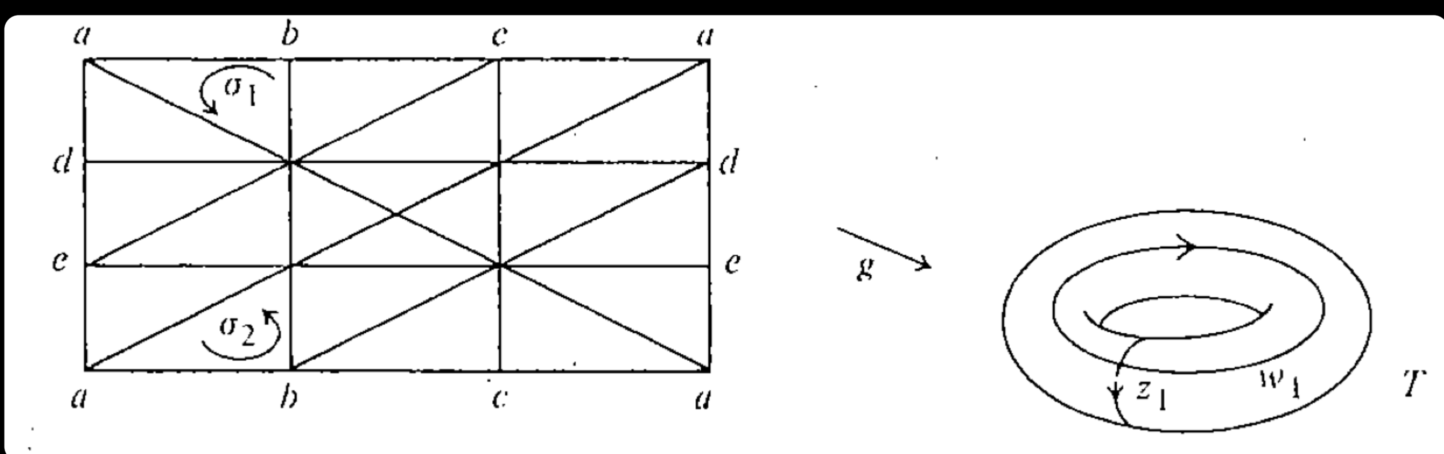
K



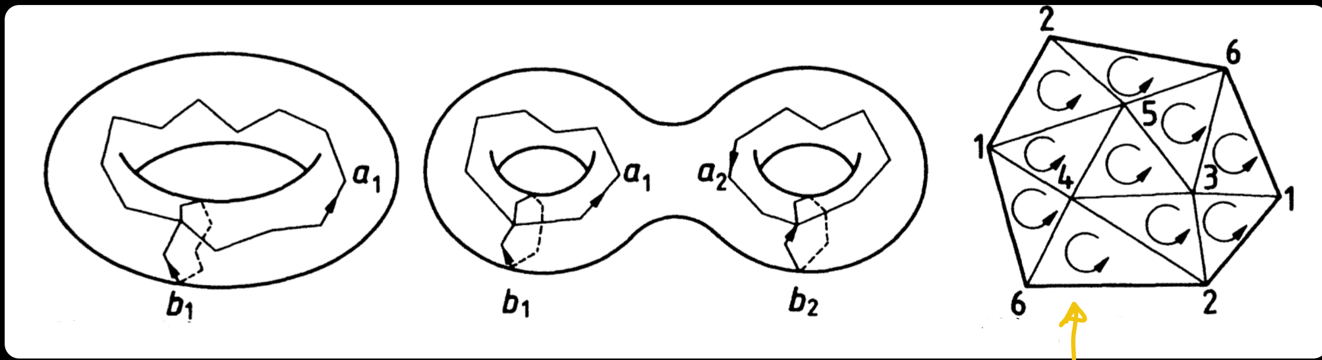
$$H_q(K) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} & q=1 \\ 0 & q \geq 2 \end{cases}$$

Ejem: T el toro

$$H_q(T) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}^2 & q=1 \\ \mathbb{Z} & q=2 \\ 0 & q \geq 3 \end{cases}$$



Ejemplos:



$\mathbb{R}P^2$

S_g = superficie orientable, género g

$$H_q(S_g) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}^{2g} & q=1 \\ \mathbb{Z} & q=2 \\ 0 & q \geq 3 \end{cases}$$

$$H_q(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/2 & q=1 \\ 0 & q \geq 2 \end{cases}$$

Ejem: K = Botella de Klein

$$H_q(K) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q=1 \\ 0 & q \geq 2 \end{cases}$$