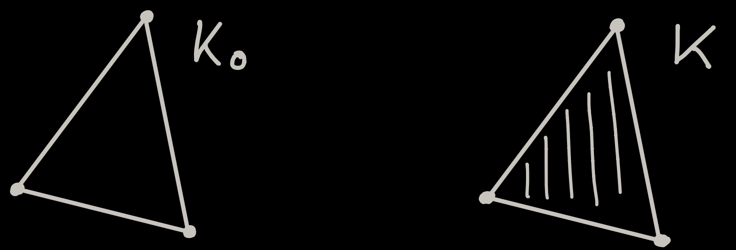
7.4 Grupos de Homología Relativos

Def: Un subconjunto $K_0 \subseteq K$ es un subcomplejo de K si K_0 es un complejo simplicial (por sí solo).



Equivalentemente: $\sigma \in K_0$ y $\tau < \sigma \Rightarrow \tau \in K_0$.

Si $K_0 \subseteq K$ es un subcomplejo, entonces:

$$C_q(K_0) \subseteq C_q(K) \quad \forall q \geq 0.$$

Def: $C_q(K, K_0) := C_q(K) \big/ C_q(K_0)$ q-cadenas relativas de K módulo K_0

Elementos: Clases laterales $\bar{c} \in C_q(K) \big/ C_q(K_0)$ donde $c \in C_q(K)$

y $\bar{c}_1 = \bar{c}_2 \iff c_1 - c_2 \in C_q(K_0)$.

Operador frontera:

$$\bar{\partial}_q : C_q(K, K_0) \longrightarrow C_{q-1}(K, K_0)$$

$$\bar{\partial}_q(\bar{c}) := \overline{\partial_q c} \quad \text{(bien definido)}$$

$$\begin{tikzcd}
C_q(K_0) \arrow[r, "i_q"] \arrow[d, "\partial_q"'] \arrow[dr, phantom, "//"] & C_q(K) \arrow[r, "j_q"] \arrow[d, "\partial_q"] \arrow[dr, phantom, "//"] & C_q(K)/C_q(K_0) \arrow[d, "\bar{\partial}_q"] \\
C_{q-1}(K_0) \arrow[r, "i_{q-1}"] & C_{q-1}(K) \arrow[r, "j_{q-1}"] & C_{q-1}(K)/C_{q-1}(K_0)
\end{tikzcd}$$

Así tenemos:

$$\cdots \xrightarrow{\bar{\partial}_{q+1}} C_q(K,K_0) \xrightarrow{\bar{\partial}_q} \cdots \xrightarrow{\bar{\partial}_2} C_1(K,K_0) \xrightarrow{\bar{\partial}_1} C_0(K,K_0) \xrightarrow{\bar{\partial}_0 = 0} 0$$

Lema: $\bar{\partial}_q \circ \bar{\partial}_{q+1} = 0$

Dem: Sea $c \in C_q(K)$ y $\bar{c} \in C_q(K,K_0)$

$$\bar{\partial}_{q+1}(\bar{c}) = \overline{\partial_{q+1}(c)}$$

$$\Rightarrow \bar{\partial}_q(\bar{\partial}_{q+1}\bar{c}) = \partial_q(\overline{\partial_{q+1}c}) = \overline{\partial_q(\partial_{q+1}c)} = 0$$

$\blacksquare$

Def: $Z_q(K,K_0) = \ker \bar{\partial}_q$ q-ciclos relativos

$B_q(K,K_0) = \operatorname{im} \bar{\partial}_{q+1}$ q-fronteras relativas

$$H_q(K,K_0) = \frac{Z_q(K,K_0)}{B_q(K,K_0)}$$ q-ésimo gpo. de homología del par (K,K_0).

Obs:

1. Sea $z \in C_q(K)$ tal que $\partial z \in C_{q-1}(K_0)$. Entonces la cadena relativa
$$\bar{z} \in C_q(K, K_0)$$
es un ciclo relativo módulo K_0.

Se suele decir que z es ciclo relativo módulo K_0.

2. El correspondiente elemento en $H_q(K, K_0)$ se denotará $\{z\}_{(K,K_0)}$.

3. Si $z' \in C_q(K)$ es otro ciclo relativo módulo K_0 entonces:
$$\{z\}_{(K,K_0)} = \{z'\}_{(K,K_0)} \iff \bar{z} - \bar{z}' \in B_q(K, K_0)$$

i.e. $\overline{z - z'} = \bar{\partial}_{q+1}(\bar{c})$ con $c \in C_{q+1}(K)$

$\overline{z - z'} = \overline{\partial_{q+1} c}$

$\iff z - z' = \partial c + c^0$ con $c^0 \in C_q(K_0)$ y $c \in C_{q+1}(K)$.

En tal caso, z y z' son homólogos módulo K_0.

Ejems:

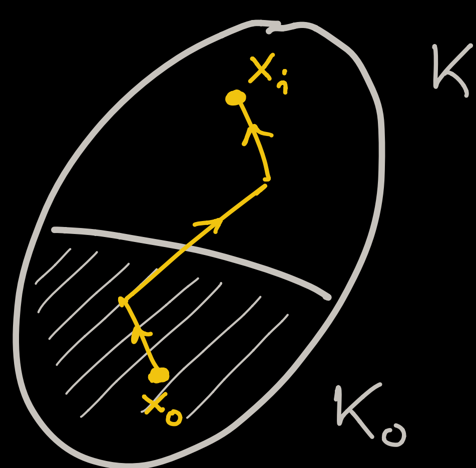
a). $\forall q \geq 0$ $H_q(K,K) = 0$ pues $C_q(K,K) = \dfrac{C_q(K)}{C_q(K)} = 0.$

b). $\forall q \geq 0$ $H_q(K,\emptyset) = H_q(K).$

c). Si $\dim(K_0) < q-1$, entonces $H_q(K,K_0) = H_q(K).$

$$\begin{tikzcd}
0 = C_{q+1}(K_0) \arrow[r, "i_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C_{q+1}(K) \arrow[r, "j_{q+1}", "\cong"'] \arrow[d, "\partial_{q+1}"'] & C_{q+1}(K,K_0) \arrow[d, "\bar{\partial}_{q+1}"] \\
0 = C_q(K_0) \arrow[r, "i_q"] \arrow[d, "\partial_q"'] & C_q(K) \arrow[r, "j_q", "\cong"'] \arrow[d, "\partial_q"'] & C_q(K,K_0) \arrow[d, "\bar{\partial}_q"] \\
0 = C_{q-1}(K_0) \arrow[r, "i_{q-1}"] & C_{q-1}(K) \arrow[r, "j_{q-1}", "\cong"'] & C_{q-1}(K,K_0)
\end{tikzcd}$$

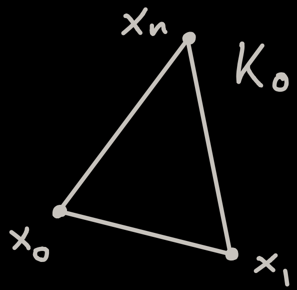
d). Si K es conexo y $\emptyset \neq K_0 \subseteq K$ subcomplejo, entonces $H_0(K,K_0) = 0.$

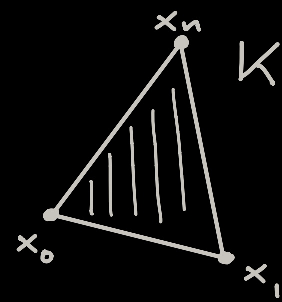


En $C_0(K,K_0)$, $\overline{x_i}$ y $\overline{x_0}$ difieren por una frontera.

$\therefore$ $\{x_i\} = \{x_0\} = 0$ en $H_0(K,K_0)$.

e). Sea σ_n n-simplejo c/ vértices $x_0, \dots, x_n$ y $K_0 = K(\partial\sigma_n) \subseteq K = K(\sigma_n)$.





Entonces:

$$C_q(K, K_0) = H_q(K, K_0) = \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Intuición: $\sigma_n / \partial\sigma_n \approx S^n$.

- Homomorfismo $j_* : H_q(K) \longrightarrow H_q(K, K_0)$

$$\begin{tikzcd} C_q(K) \arrow[r, "j_q"] \arrow[d, "\partial_q"'] \arrow[dr, phantom, "/\!/"] & C_q(K, K_0) \arrow[d, "\overline{\partial}_q"] \\ C_{q-1}(K) \arrow[r, "j_{q-1}"] & C_{q-1}(K, K_0) \end{tikzcd}$$

$\therefore$ j_q manda ciclos en ciclos.

Definimos: $j_* : H_q(K) \longrightarrow H_q(K, K_0)$

$$\{z\}_K \longmapsto \{\overline{z}\}_{(K, K_0)}$$

(Bien definido, homomorfismo)

- Homomorfismo de conexión: $\partial_*: H_q(K,K_0) \longrightarrow H_{q-1}(K_0)$

$$
\begin{tikzcd}
 & z \arrow[r, mapsto] \arrow[dddd, bend left] & \bar{z} \\
C_q(K_0) \arrow[r, "i_q"] \arrow[d, "\partial_q"'] \arrow[dr, phantom, "="] & C_q(K) \arrow[r, "j_q"] \arrow[d, "\partial_q"'] \arrow[dr, phantom, "="] & C_q(K,K_0) \arrow[d, "\bar{\partial}_q"] \\
C_{q-1}(K_0) \arrow[r, "i_{q-1}"] & C_{q-1}(K) \arrow[r, "j_{q-1}"] & C_{q-1}(K,K_0) \\
\partial_q(z) \arrow[r, mapsto] & \partial_q(z) & \bar{0}
\end{tikzcd}
$$

Definimos: $\partial_* : H_q(K,K_0) \longrightarrow H_{q-1}(K_0)$

$$\{z\}_{(K,K_0)} \longmapsto \{\partial z\}_{K_0}$$

Ejercicio: ∂_* bien definido y es homomorfismo.

Def: La sucesión de gpos. y homomorfismos

$$\cdots \xrightarrow{j_*} H_{q+1}(K,K_0) \xrightarrow{\partial_*} \underbrace{H_q(K_0) \xrightarrow{i_*} H_q(K) \xrightarrow{j_*} H_q(K,K_0)}_{\text{patrón en dim. } q} \xrightarrow{\partial_*} H_{q-1}(K_0) \longrightarrow \cdots$$

es la <u>sucesión exacta larga</u> (S.E.L.) en homología de la pareja (K,K_0).

"Sucesión exacta" significa

a). $\mathrm{im}\{\partial_*: H_{q+1}(K,K_0) \to H_q(K_0)\} = \ker\{i_*: H_q(K_0) \to H_q(K)\}$

b). $\mathrm{im}\{i_*: H_q(K_0) \to H_q(K)\} = \ker\{j_*: H_q(K) \to H_q(K,K_0)\}$

c). $\mathrm{im}\{j_*: H_q(K) \to H_q(K,K_0)\} = \ker\{\partial_*: H_q(K,K_0) \to H_{q-1}(K_0)\}$

Dem: Ejercicio.

Tma: Sea $\varphi:(K,K_0) \to (L,L_0)$ mapeo simplicial entre complejos simpliciales: $\varphi: K \to L$ y $\varphi(K_0) \subseteq L_0$.

a). $\varphi: K \to L$ induce $\varphi_*: H_q(K) \to H_q(L)$.

b). $\varphi|_{K_0}: K_0 \to L_0$ induce $(\varphi|_{K_0})_*: H_q(K_0) \to H_q(L_0)$.

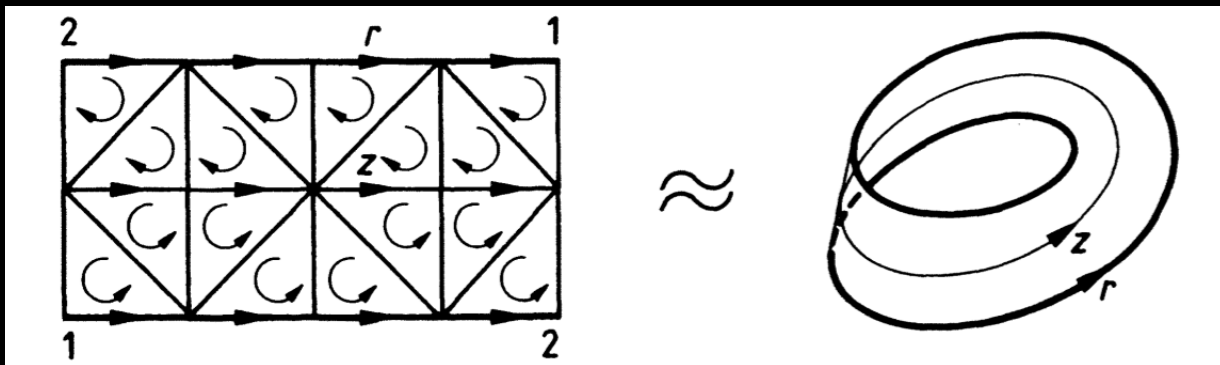
c). La función $\{z\}_{(K,K_0)} \longmapsto \{\varphi_\circ(z)\}_{(L,L_0)}$ define un homomorfismo

$$\varphi_*: H_q(K,K_0) \longrightarrow H_q(L,L_0)$$

y el sig. diagrama es conmutativo:

$$\begin{tikzcd}
\cdots \arrow[r, "j_*"] & H_{q+1}(K,K_0) \arrow[r, "\partial_*"] \arrow[d, "\varphi_*"] & H_q(K_0) \arrow[r, "i_*"] \arrow[d, "(\varphi|_{K_0})_*"] & H_q(K) \arrow[r, "j_*"] \arrow[d, "\varphi_*"] & H_q(K,K_0) \arrow[r, "\partial_*"] \arrow[d, "\varphi_*"] & \cdots \\
\cdots \arrow[r, "j_*"] & H_{q+1}(L,L_0) \arrow[r, "\partial_*"] & H_q(L_0) \arrow[r, "i_*"] & H_q(L) \arrow[r, "j_*"] & H_q(L,L_0) \arrow[r, "\partial_*"] & \cdots
\end{tikzcd}$$

Ejem: La figura sig. representa a una banda de Möbius (triangulada) M



$\partial M =$ circunferencia triangulada $\approx S^1$

$H_1(\partial M) \cong \mathbb{Z}$ generado por $\{r\}_{\partial M}$

$H_1(M) \cong \mathbb{Z}$ generado por $\{z\}_M$

S.E.L. de la pareja $(M, \partial M)$:

$$\begin{tikzcd}
H_1(\partial M) \arrow[r, "i_*"] \arrow[d, equal] & H_1(M) \arrow[r, "j_*"] \arrow[d, equal] & H_1(M,\partial M) \arrow[r, "\partial_*"] & H_0(\partial M) \arrow[r, "i_*", "\cong"'] & H_0(M) \\
\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} & & &
\end{tikzcd}$$

Notemos que: i_* es isomorfismo.

$$\Rightarrow 0 = \ker i_* = \operatorname{im} \partial_*$$

$$\Rightarrow H_1(M, \partial M) = \ker \partial_* = \operatorname{im} j_*$$

i.e. j_* es sobre.

$$\therefore H_1(M,\partial M) \cong H_1(M)/\ker j_* = H_1(M)/\operatorname{im} i_* \cong \mathbb{Z}/2.$$

$i_* : H_1(\partial M) \longrightarrow H_1(M)$ es multiplicación por 2 :

c_2 = suma de todos los triángulos orientados de M.

$\partial c_2 = r - 2z$

$\therefore \ \{r\} = 2\{z\}$ en $H_1(M)$

$\therefore \ i_* \{r\}_{\partial M} = 2\{z\}_M$.