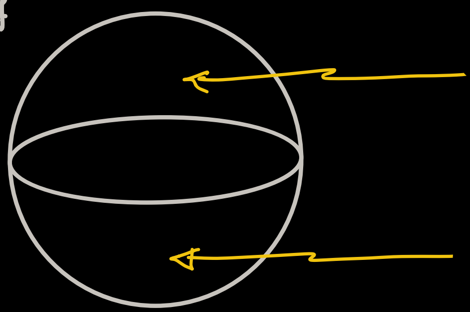
Ejemplo: La homología de $\mathbb{R}P^n$

Consideremos las esferas $S^0 \subseteq S^1 \subseteq \dots \subseteq S^n$
y $\forall\, q = 0, 1, \dots, n$ las q-celdas:

S^q

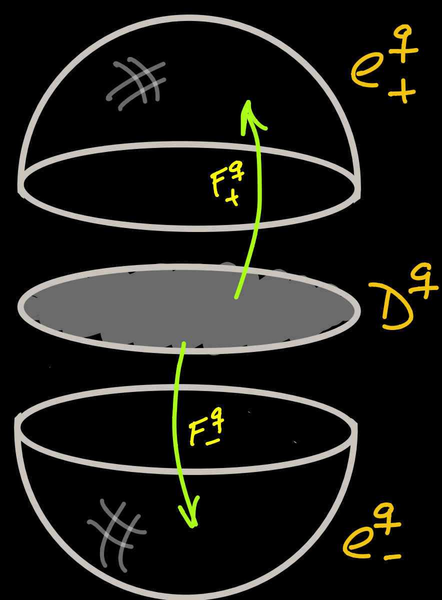


$$e^q_+ = D^q_+ = \{ x \in S^q \mid x_{q+1} > 0 \}$$

$$e^q_- = D^q_- = \{ x \in S^q \mid x_{q+1} < 0 \}$$

Entonces: $S^n = (e^0_+ \cup e^0_-) \cup \dots \cup (e^n_+ \cup e^n_-)$.

Mapeos característicos



$$F^q_\pm : D^q \longrightarrow S^q$$

$$F^q_\pm(x) = (x_1, \dots, x_q, \pm\sqrt{1 - |x|^2})$$

Celdas orientadas:

$$(F^q_\pm)_* : H_q(D^q, S^{q-1}) \longrightarrow H_q(S^q, S^{q-1}) = \underbrace{C_q(S^n)}_{\text{cadenas celulares}}$$

$$\{D^q\} \longmapsto e^q_\pm$$

Abuso de notación $\begin{cases} e_+^q := (F_+^q)_*(\{D^q\}) \\ e_-^q := (F_-^q)_*(\{D^q\}) \end{cases}$ Base para $C_q(S^n)$

Mapeo antipodal: $\begin{aligned} a : S^n &\longrightarrow S^n \\ x &\longmapsto -x \end{aligned}$

- Geométricamente: $a : e_+^q \xrightarrow{\approx} e_-^q$

- Algebraicamente: $a : (S^q, S^{q-1}) \xrightarrow{\approx} (S^q, S^{q-1})$

$$a_* : H_q(S^q, S^{q-1}) \xrightarrow{\cong} H_q(S^q, S^{q-1})$$

i.e. $a_* : C_q(S^n) \xrightarrow{\cong} C_q(S^n)$

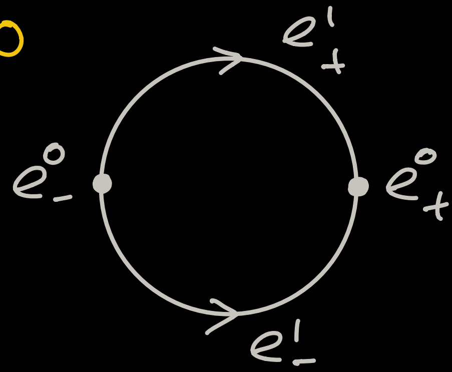
es un isomorfismo.

Lema:

a) $a_*(e_+^q) = (-1)^q e_-^q$

b) $\partial e_+^{q+1} = \partial e_-^{q+1} = \pm(e_+^q - e_-^q)$.

Dem: $q=0$



(a) $a_*(e^0_+) = e^0_-$.

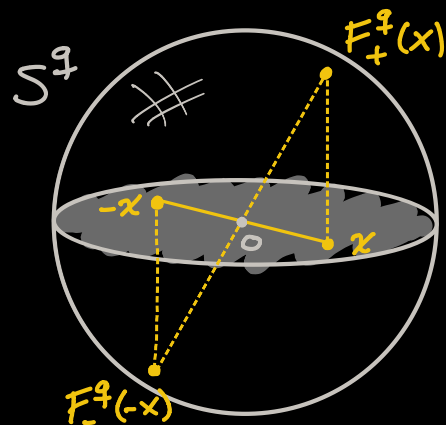
(b) $\partial e^1_+ = e^0_+ - e^0_-$

$\partial e^1_- = e^0_- - e^0_-$.

Caso $q \geq 1$:

a). Sea $g: D^q \to D^q$, $g(x) = -x$. Notemos que:

$$\begin{tikzcd} D^q \arrow[r, "F^q_+"] \arrow[d, "g"'] \arrow[dr, phantom, "="] & S^q \arrow[d, "a"] \\ D^q \arrow[r, "F^q_-"] & S^q \end{tikzcd}$$



$$\begin{tikzcd} H_q(D^q, S^{q-1}) \arrow[r, "F^q_{+*}"] \arrow[d, "g_*"'] \arrow[dr, phantom, "="] & H_q(S^q, S^{q-1}) \arrow[d, "a_*"] \\ H_q(D^q, S^{q-1}) \arrow[r, "F^q_{-*}"] & H_q(S^q, S^{q-1}) \end{tikzcd}$$

$$a_*(F^q_+)_*\{D^q\} = (F^q_-)_* g_*\{D^q\}$$

$g|_{S^{q-1}}$ = mapeo antipodal $S^{q-1} \to S^{q-1}$

$$\therefore \quad a_*(e^q_+) = (-1)^q e^q_-$$

b). Frontera celular:

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \\
C_{q+1}(X) = H_{q+1}(X^{q+1}, X^{q}) \arrow[rr, "\partial_*"] \arrow[rrdd, bend right, "\partial^{cel}_{q+1}"'] & & H_q(X^q) \arrow[r] \arrow[d, "j_*"] & \dots \\
 & & H_q(X^q, X^{q-1}) \arrow[d, equal] & \\
 & & C_q(X) &
\end{tikzcd}$$

Caso $X = S^n$:

$$C_{q+1}(S^n) = H_{q+1}(S^{q+1}, S^q) \cong \mathbb{Z}^2 = \langle e_+^{q+1}, e_-^{q+1} \rangle$$

$$C_q(S^n) = H_q(S^q, S^{q-1}) \cong \mathbb{Z}^2 = \langle e_+^{q}, e_-^{q} \rangle$$

$$\{D^q\} \longmapsto \{S^q\}$$

$F_+^{q+1}\big|_{S^q} = id_{S^q}$

$$\begin{tikzcd}
{} \arrow[r] & H_{q+1}(D^{q+1}, S^q) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "(F_+^q)_*"'] \arrow[dr, phantom, "//"] & H_q(S^q) \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & H_{q+1}(S^{q+1}, S^q) \arrow[r, "\partial_*"] & H_q(S^q) \arrow[r] \arrow[d, "j_*"] & {} \\
 & & H_q(S^q, S^{q-1}) &
\end{tikzcd}$$

SEL's de las parejas

Luego $\partial_{q+1} : C_{q+1}(S^n) \longrightarrow C_q(S^n)$

$$\begin{array}{ccc} C_{q+1}(S^n) & \longrightarrow & C_q(S^n) \\ \| & & \| \\ \mathbb{Z}^2 & & \mathbb{Z}^2 \end{array}$$

está dada por: $\partial_{q+1}(e^{q+1}_+) = j_*(\{S^q\})$

(similarmente) $\partial_{q+1}(e^{q+1}_-) = j_*(\{S^q\})$

Pero en la suc. exacta

$$\begin{array}{ccccc} H_q(S^q) & \xrightarrow{j_*} & H_q(S^q, S^{q-1}) & \xrightarrow{\partial_*} & H_{q-1}(S^{q-1}) \\ & & \cong & & \cong \\ & & \langle e^q_+, e^q_- \rangle & & \mathbb{Z} \end{array}$$

- $\ker \partial_* = \langle e^q_+ - e^q_- \rangle \cong \mathbb{Z}$

Subgpo. generado

y

- $\ker \partial_* = \mathrm{im}\, j_* = \langle j_*(\{S^n\}) \rangle$

$\therefore \quad j_*(\{S^q\}) = \pm(e^q_+ - e^q_-)$

$\Rightarrow \quad \partial^{cel}_{q+1}(e^{q+1}_+) = \partial^{cel}_{q+1}(e^{q+1}_-) = \pm(e^q_+ - e^q_-).$ ▨

Estrategia: Comparar las cadenas celulares de S^n y $\mathbb{RP}^n$.

Recordemos $\mathbb{RP}^n = S^n / x \sim \pm x$

- $p: S^n \longrightarrow \mathbb{RP}^n$ proyección canónica.
- p manda $S^q \subseteq S^n$ en $\mathbb{RP}^q \subseteq \mathbb{RP}^n$.
- $\mathbb{RP}^n$ es un CW-complejo con celdas

$$e^q = p(e^q_+) = p(e^q_-), \quad q = 0, 1, \dots, n.$$

Mapeo característico para e^q: (elección!)

$$D^q \xrightarrow{F^q_+} S^q \xrightarrow{p} \mathbb{RP}^q$$

$$\begin{tikzcd}
H_q(D^q, S^{q-1}) \arrow[r, "F^q_{+*}"] & H_q(S^q, S^{q-1}) \arrow[r, "p_*"] \arrow[d, equal] & H_q(\mathbb{RP}^q, \mathbb{RP}^{q-1}) \arrow[d, equal] \\
 & C_q(S^n) & C_q(\mathbb{RP}^n) \\
\{D^q\} \arrow[r, mapsto] & e^q_+ \arrow[r, mapsto] & e^q
\end{tikzcd}$$

q-celda orientada en $\mathbb{RP}^q$

i.e. $\quad p_*(e^q_+) =: e^q$

Notemos que:

$$\begin{tikzcd} S^q \arrow[rr, "a"] \arrow[dr, "p"'] & \arrow[d, phantom, "="] & S^q \arrow[dl, "p"] \\ & \mathbb{R}P^q & \end{tikzcd} \Longrightarrow \begin{tikzcd} H_q(S^q, S^{q-1}) \arrow[rr, "a_*"] \arrow[dr, "p_*"'] & \arrow[d, phantom, "="] & H_q(S^q, S^{q-1}) \arrow[dl, "p_*"] \\ & H_q(\mathbb{R}P^q, \mathbb{R}P^{q-1}) & \end{tikzcd}$$

$$\begin{aligned} p_*(e_+^q) &= p_* a_*(e_+^q) \\ &= p_*\left((-1)^q e_-^q\right) \end{aligned}$$

$$\therefore \; p_*(e_-^q) = (-1)^q e^q$$

Comparemos los complejos de cadenas celulares:

$$\begin{tikzcd} & e_+^{q+1} \arrow[rr, mapsto] \arrow[dd, bend right, mapsto] & & \pm(e_+^q - e_-^q) \arrow[dd, bend left, mapsto] & \\ \cdots \arrow[r] & C_{q+1}(S^n) \arrow[r, "\partial_{q+1}"] \arrow[d, "p_*"'] \arrow[dr, phantom, "="] & C_q(S^n) \arrow[r] \arrow[d, "p_*"] & \cdots & \\ \cdots \arrow[r] & C_{q+1}(\mathbb{R}P^n) \arrow[r, "\partial_{q+1}"] & C_q(\mathbb{R}P^n) \arrow[r] & \cdots & \\ & e^{q+1} \arrow[rr, mapsto] & & \pm[e^q - (-1)^q e^q] & \end{tikzcd}$$

$$= \pm[1 - (-1)^q]\, e^q$$

$$\partial : C_{q+1}(\mathbb{RP}^n) \longrightarrow C_q(\mathbb{RP}^n) \qquad \text{si } q \text{ es impar}$$
$$e^{q+1} \longmapsto \pm 2e^q \qquad (q \geq 1)$$

$$\partial : C_{q+1}(\mathbb{RP}^n) \longrightarrow C_q(\mathbb{RP}^n) \qquad \text{si } q \text{ es par.}$$
$$e^{q+1} \longmapsto 0 \qquad (q \geq 0)$$

Complejo de cadenas celulares de $\mathbb{RP}^n$:

$$0 \to C_n(\mathbb{RP}^n) \to \cdots \to C_1(\mathbb{RP}^n) \to C_0(\mathbb{RP}^n) \to 0$$

$$0 \to \underset{\substack{\text{dim}\\ n}}{\mathbb{Z}} \to \cdots \xrightarrow{\times 0} \underset{\substack{\text{dim}\\ 2}}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\substack{\text{dim}\\ 1}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\text{dim}\\ 0}}{\mathbb{Z}} \to 0$$

$$H_q(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}_2 & q \text{ impar}, \ 1 \leq q \leq n-1 \\ 0 & q \text{ par}, \ 2 \leq q \leq n \\ \mathbb{Z} & q = n, \ n \text{ impar} \\ 0 & q > n \end{cases}$$

Ejems:

$n=2$

$$\cdots \to 0 \xrightarrow{0} \underset{\text{dim } 2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\text{dim } 1}{\mathbb{Z}} \xrightarrow{0} \underset{\text{dim } 0}{\mathbb{Z}} \to 0$$

$$H_q(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}_2 & q=1 \\ 0 & q=2 \\ 0 & \text{otro caso} \end{cases}$$

$n=3$

$$\cdots \to 0 \xrightarrow{0} \underset{\text{dim } 3}{\mathbb{Z}} \xrightarrow{0} \underset{\text{dim } 2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\text{dim } 1}{\mathbb{Z}} \xrightarrow{0} \underset{\text{dim } 0}{\mathbb{Z}} \to 0$$

$$H_q(\mathbb{R}P^3) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}_2 & q=1 \\ 0 & q=2 \\ \mathbb{Z} & q=3 \\ 0 & \text{otro caso.} \end{cases}$$

$n=4$

$$\cdots \quad 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\begin{array}{ccccc} \text{dim } 4 & \text{dim } 3 & \text{dim } 2 & \text{dim } 1 & \text{dim } 0 \end{array}$$

$$H_q(\mathbb{R}P^4) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}_2 & q=1 \\ 0 & q=2 \\ \mathbb{Z}_2 & q=3 \\ 0 & \text{otro caso} \end{cases}.$$